MODULI SPACES OF BRANCHED COVERS OF VEECH SURFACES I: d-SYMMETRIC DIFFERENTIALS

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ABSTRACT. We give a description of asymptotic quadratic growth rates for geodesic segments on covers of Veech surfaces in terms of the *modular fiber* parameterizing coverings of a fixed Veech surface. To make the paper self contained we derive the necessary asymptotic formulas from the Gutkin-Judge formula. As an application of the method we define and analyze d-symmetric elliptic differentials and their modular fibers $\mathscr{F}_d^{sym}$. For given genus g, g-symmetric elliptic differentials (with fixed base lattice) provide a 2-dimensional family of translation surfaces. We calculate several asymptotic constants, to establish their dependence on the translation geometry of $\mathscr{F}_d^{sym}$ and their sensitivity as $\mathrm{SL}_2(\mathbb{Z})$-orbit invariants.

CONTENTS

1. INTRODUCTION

During the last decade major progress in calculating asymptotic quadratic growth rates of geodesic segments on translation surfaces has been made. After introducing an asymptotic formula for a special type of surfaces (see [V2, V3]), now called lattice-, or Veech-surfaces, Veech found a fairly general integral-formula [V4] relating the asymptotic constants of a translation surface to the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of the surface in the moduli space of translation surfaces. To evaluate the so called *Siegel-Veech formula* for a

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particular translation surface one needs to calculate the volume of the closure of the surfaces $\mathrm{SL}_2(\mathbb{R})$-orbit in moduli space, which in general is not known to be well-defined, since presently there are no general structure theorems for $\mathrm{SL}_2(\mathbb{R})$-orbit classification. First assuming the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of a surface is a complex manifold, one can rewrite part of the Siegel-Veech constants in terms of volume of this orbit closure. To complete the evaluation of Siegel-Veech constants one has to know the orbit(s) of the surface with respect to the action (of conjugates) of the unipotent group

$$\mathscr{U} := \{\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] : \ t \in \mathbb{R}\} \subset \mathrm{SL}_2(\mathbb{R}).$$

Specifying an $\mathscr{U}$-orbit closure classification is known as the *Ratner problem* for the moduli space of (translation) surfaces (see [EMZ]). Improvement of Veechs work by Eskin and Masur [EM98] says that the *generic* surface in a stratum admits Siegel-Veech constants. Essentially a "generic" surface has the whole stratum as its $\mathrm{SL}_2(\mathbb{R})$-orbit closure. Strata are complex manifolds and certain Siegel-Veech constants for "generic" surfaces in a given stratum have been calculated in [EM98, EMZ].

This leaves the *fundamental problem* to tell, if a given surface actually belongs to the *generic set*. Jet another difficulty is the *complexity* arising in volume calculations for "spaces" of *high genus surfaces*. Moduli space volumes rely on counting torus coverings and have been studied in [EMZ] using results of [EO].

A class of examples where one understands all $\mathrm{SL}_2(\mathbb{R})$ (and $\mathscr{U}$) orbit closures are spaces of torus-covers. In fact spaces of torus-covers cover a homogeneous space themselves, therefore we have a *Ratner-Theorem* with respect to the action of the unipotent group $\mathscr{U}$ on spaces of torus-covers. Recently Eskin, Marklof and Morris-Witte established a Ratner Theorem for spaces of coverings of Veech surfaces which are not tori (see [EMM]).

For genus 2 torus-covers generic Siegel-Veech constants were calculated in [EMS].

So far the literature does not provide an applicable method to calculate Siegel-Veech constants for surfaces in arbitrary high genus, avoiding the above mentioned difficulties of the general approach. There is motivation to have such a method partly related to polygonal billiards, but also to study the inner structure of series of asymptotic constants as found in [EMS]. In this paper we develop a method to calculate Siegel-Veech constants [S3] for moduli spaces of coverings of Veech-surfaces and express the constants in terms of the translation geometry of the modular fiber parameterizing coverings of a fixed Veech surface. Finally we define and describe a 2-dimensional (fiber-) space of torus-covers in any genus, including an evaluation of Siegel-Veech constants for all surfaces parameterized by these modular fibers.

Recall that a translation surface is a Riemann surface X together with an atlas whose chart changes are translations in $\mathbb{C}$. Equivalently one can consider pairs (X, ω), where X is a Riemann surface and $\omega \in \Omega(X)$ a non-trivial

holomorphic 1-form on X. A translation atlas allows to glue together local the local 1-forms dz, obtaining a global holomorphic 1-form ω on X. The induced metric has finitely many *singular points*, or *cone points*, which are located in $Z(\omega)$, the set where ω vanishes. This allows to consider the set $SC_X(p_1, p_2)$ of all geodesic segments connecting two given cone points $p_1, p_2 \in Z(\omega)$, *not containing any other cone point.* Goedesic segments with this property are usually called *saddle connections.*

Closed geodesics on (X, ω) not containing cone points typically appear in (maximal) families of parallel geodesics having the same length, say w. Any maximal family covers an open set on X isometric to a cylinder $\mathscr{C} = \mathbb{R}/w\mathbb{Z} \times (0, h)$, the positive number h is called *height* of the cylinder and $w = w(\mathscr{C})$ is its *(core) width.* The images of the closed geodesics under this isometry are *horizontal loops* $\mathbb{R}/w\mathbb{Z} \times \{l\} \subset \mathscr{C}$.

To specify the growth rate problems take the set of maximal cylinders Cyl_X on X or one of the sets $SC_X(p_1, p_2)$ and consider the counting function $\mathbb{R}_+ \to \mathbb{N}$

$$T \mapsto N(Cyl_X, T) := |\{\mathscr{C} \in Cyl_X : w(\mathscr{C}) < T\}|.$$

The following result is fundamental (see [M1, M2] and [EM98] for a recent version):

Theorem [Masur] *There are constants c_{min}, c_{max} such that*

$$0 < c_{min} T^2 < N(Cyl_X, T) < c_{max} T^2, \quad \textit{whenever} \;\; T >> 0.$$

A similar statement holds for the function $N(SC_X(p_1, p_2), T)$ counting saddle connections between $p_1 \in X$ and $p_2 \in X$ shorter than T. In case the *space* $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)}$ obeys a Ratner Theorem, Masur's Theorem upgrades to

$$\pi \cdot c_{cyl}(X) = \lim_{T \to \infty} \frac{N(Cyl_X, T)}{T^2}.$$

We recall that the $\mathrm{SL}_2(\mathbb{R})$-action on translation surfaces is defined as follows:

Take a natural chart $\zeta(p) = \int_{p_0}^{p} \omega$ around $p_0 \in X \backslash Z(\omega)$ and simply postcompose this chart using $A \in \mathrm{SL}_2(\mathbb{R})$ acting real linear on $\mathbb{R}^2 \cong \mathbb{C}$. Doing this for all natural charts gives a new translation structure $(\widetilde{X}, \widetilde{\omega}) = A \cdot (X, \omega)$. Obviously the numbers and orders of the zeros of $\widetilde{\omega}$ agree with the ones of ω. Since the $\mathrm{SL}_2(\mathbb{R})$-action descends to an action on the moduli space of translation structures, the above implies that the $\mathrm{SL}_2(\mathbb{R})$-action preserves sets of Abelian differentials with given singularity pattern, so called *strata.* Strata are complex orbifolds [V1] and $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)}$ is typically (for the generic surface) the whole stratum, but sometimes just a subset (of shape which has yet to be described).

We are interested in Abelian differentials (X, ω) which are branched covers of tori.

Elliptic differentials. A *lattice* $\Lambda \subset \mathbb{R}^2 \cong \mathbb{C}$ is a subgroup such that $\mathbb{R}^2/\Lambda$ is a torus. The pair (X, ω) is called *elliptic differential*, if there is a lattice Λ and holomorphic map $\pi : X \to \mathbb{C}/\Lambda$, such that

$$\omega = \pi^* dz.$$

Throughout the paper we assume lattices denoted with Λ are *unimodular*, i.e.:

$$\int_{\mathbb{C}/\Lambda} dx \wedge dy = 1.$$

However for our purpose a slightly extended notion of elliptic differentials is appropriate to cover *degenerated differentials*.

Degenerated elliptic differentials. An *elliptic differential* $(X, p_1, \dots, p_n, \omega)$ consists of the following data:

- *a topological space* X *and a set of points* $\{p_1, ...p_n\} \in X$
- *a Riemann surface structure on* $X - \{p_1, ...p_n\}$
- *a branched covering* $p : X - \{p_1, ...p_n\} \longrightarrow \mathbb{C}/\Lambda$
- *an induced differential* $\omega = p^* dz$ *on* $X - \{p_1, ...p_n\}$.

We call an elliptic differential *regular*, if $(X, p_1, ..., p_n, \omega)$ extends to a differential on X, otherwise we call $(X, p_1, ..., p_n, \omega)$ *degenerated*. Note that this notion includes for example unions of tori over the same base lattice, identified in finitely many points. This class of degenerated elliptic differentials appears naturally in the compactification of modular fibers. The degenerated differentials are just *stable curves* equipped with a holomorphic differential away from finitely many *singular* points.

Connected spaces of elliptic differentials $\mathscr{E}$, which are closed under the action of $\mathrm{SL}_2(\mathbb{R})$ admit a *fiber bundle* structure:

$$\mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}). \tag{1}$$

Here the fiber $\mathscr{F}$ is the set of all torus covers in $\mathscr{E}$ which cover a particular torus $\mathbb{C}/\Lambda$. Usually we consider the fiber over $\Lambda = \mathbb{Z}^2$ which causes no restrictions of generality. It is easy to see, that the *lattice of absolute periods* of τ, i.e.

$$\mathrm{Per}(\tau) := \left\{ \int_\gamma \tau : [\gamma] \in H_1(Y; \mathbb{Z}) \right\} \subset \mathbb{C},$$

equals Λ for all $(Y, \tau) \in \mathscr{F}$. We will prove the following special version of a statement in [S2]:

Theorem 1. *Given an elliptic differential* (Y, τ) *with* $\mathrm{Per}(\tau) = \mathbb{Z}^2$ *and exactly two cone points* p_0, p_1. *Assume* $\pi(p_1) - \pi(p_0) \notin \mathbb{Q}^2/\mathbb{Z}^2$, *i.e. is not torsion with respect to the natural translation cover* $\pi : Y \to Y/\mathrm{Per}(\tau)$. *Then* $\mathscr{F}_\tau := \overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (Y, \tau)}$ *admits a natural structure as an elliptic differential* $(\mathscr{F}_\tau, \omega_\tau)$ *with a lattice group of affine homeomorphisms:*

$$\mathrm{D}\,\mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau) \cong \mathrm{SL}(\mathbb{C}/\mathrm{Per}(\tau), dz) \cong \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

In addition $\theta \in S^1$ *is a periodic direction on* (X,ω), *if and only if it is periodic on* $\mathscr{F}_\tau$. $\mathscr{F}_\tau$ *decomposes into* $n_{\mathscr{F}}$ *maximal open cylinders* $\mathscr{C}_1, ..., \mathscr{C}_{n_{\mathscr{F}}}$, *with top boundary components* $\partial^{top}\mathscr{C}_1, ..., \partial^{top}\mathscr{C}_{n_{\mathscr{F}}}$.

The cylinder decomposition $\mathscr{C}_{i,1}(\alpha), ..., \mathscr{C}_{i,n_i}(\alpha)$ *of any* $(S,\alpha) \in \mathscr{C}_i$ *in direction* θ *is the same, in the sense that the foliation* $\mathscr{F}_\theta(S,\alpha)$ *contains* n_i *cylinders having width* $w_{i,j} = w(\mathscr{C}_{i,j}(\alpha))$ *independent of* $(S,\alpha) \in \mathscr{C}_i$.

Remark 1. The space $\mathscr{F}_\tau$ might not be connected, for this reason we use its affine group rather than assigning a "Veech group" to it. For example think of an arithmetic torus cover (Y,τ), then the modular fiber

$$\overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (Y,\tau)} = \mathscr{F}_\tau.$$

is a finite union of Veech surfaces with Veech group a sub-group of $\mathrm{SL}_2(\mathbb{Z})$. We then say the affine group of this union of surfaces, i.e. the modular fiber $\mathscr{F}_\tau$, is $\mathrm{Aff}^+(\mathscr{F}_\tau)$ which in turn has linear part $\mathrm{D}\,\mathrm{Aff}^+(\mathscr{F}_\tau) = \mathrm{SL}_2(\mathbb{Z})$. More general, if we start with a torus cover (Y,τ) and obtain a 2-dimensional modular fiber

$$\overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (Y,\tau)} = \mathscr{F}_\tau = \overline{\mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau) \cdot (Y,\tau)},$$

having affine group $\mathrm{Aff}^+(\mathscr{F}_\tau,\omega_\tau)$ with linear part $\mathrm{D}\,\mathrm{Aff}^+(\mathscr{F}_\tau,\omega_\tau) \cong \mathrm{SL}_2(\mathbb{Z})$. It is known (see [GHS], [EMM]) that the $\mathrm{Aff}^+(\mathscr{F}_\tau,\omega_\tau)$-orbit of a non-periodic "point" $(Y,\tau) \in \mathscr{F}_\tau$ is equally distributed in $\mathscr{F}_\tau$ (w.r.t. Lebesque measure).

Remark 2. Theorem 1 can be generalized to non-arithmetic surfaces. Modular fibers $(\mathscr{F}_\tau,\omega_\tau)$ are surfaces (or spaces) which should be studied for itself. The modular fibers of genus 2 torus-coverings for example provides a family of arithmetic surfaces with unbounded genus, but yet accessible structural properties, see [S3]. Subspaces of (higher dimensional) modular fibers which are Veech-group invariant and 2-dimensional are studied in [HST].

In this paper we study the easiest case of modular fibers: when the modular fiber is a torus itself.

Asymptotic constants via modular fibers. From now on, whenever we write down a modular fiber as a pair $(\mathscr{F},\omega_{\mathscr{F}})$, i.e. equipped with an Abelian differential, we assume it has dimension 2.

Theorem 2. [S3] *Assume* $(\mathscr{F}_\tau,\omega_\tau)$ *is the modular fiber of a space of elliptic differentials which are translation covers of* $(\mathbb{R}^2/\mathbb{Z}^2, dz)$. *Denote the maximal horizontal cylinders in* $\mathscr{F}_h(\mathscr{F}_\tau,\omega_\tau)$ *by* $\mathscr{C}_1, \dots, \mathscr{C}_{n_{\mathscr{F}}}$. *Further assume the* n_i *maximal cylinders in the horizontal foliation of* $(S,\alpha) \in \mathscr{C}_i$ *have width* $w_{i,1}, ..., w_{i,n_i}$. *If* $(S,\alpha) \in \mathscr{F}_\tau$ *has infinite* $\mathrm{Aff}^+(\mathscr{F}_\tau,\omega_\tau)$ *orbit, then it admits the Siegel-Veech constant*

$$c_{cyl}(\alpha) = \frac{1}{\mathrm{area}(\mathscr{F}_\tau)} \sum_{i=1}^{n_{\mathscr{F}}} \sum_{k=1}^{n_i} \frac{\mathrm{area}(\mathscr{C}_i)}{w_{i,k}^2} \tag{3}$$

for maximal cylinders. If the orbit

$$\mathscr{O}_\alpha := \mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau) \cdot (S, \alpha) \subset \mathscr{F}_\tau$$

is finite, we have

$$c_{cyl}(\alpha) = \frac{1}{|\mathscr{O}_\alpha|} \sum_{i=1}^{n_{\mathscr{F}}} \left(\sum_{k=1}^{n_i} \frac{|\mathscr{O}_\alpha \cap \mathscr{C}_i|}{w_{i,k}^2} + \sum_{k=1}^{m_i} \frac{|\mathscr{O}_\alpha \cap \partial^{top} \mathscr{C}_i|}{w_{i,k}^2} \right). \tag{4}$$

The Theorem expresses Siegel-Veech constants as quantities depending on the horizontal foliation of the modular fiber $(\mathscr{F}_\tau, \omega_\tau)$. Note that each leaf $\mathscr{L} \in \mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$ has a finite $\mathscr{U} \cap \mathrm{SL}_2(\mathbb{Z})$ orbit. Looking at the $\mathscr{U}$ orbit of $\mathscr{L}$ in $\mathscr{E}$, we see either a disjoint union of cylinders, if $\mathscr{L}$ is a saddle connection, or a disjoint union of 2-tori in $\mathscr{E}$, if $\mathscr{L}$ is a regular loop. That shows, the $\mathscr{U}$ orbit closures (of (S, α)), might be much smaller than the $\mathrm{SL}_2(\mathbb{Z})$-orbits (of (S, α)).

Saddle connections. For saddle connections one obtains a very similar formula involving the saddle connections in $\mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$. Indeed given a saddle connection s on a finite $\mathrm{SL}_2(\mathbb{Z})$ orbit surface $(S, \alpha) \in \mathscr{F}_\tau$ with two cone points we can always find an $A \in \mathrm{SL}_2(\mathbb{Z})$, such that on $A \cdot (S, \alpha) \in \mathscr{F}_\tau$ the image of s, i.e. $A \cdot s$, is horizontal. Restriction to the two cone point situation implies, that for each horizontal saddle connection $s \in \mathscr{F}_h(S, \alpha)$ which connects one and the same cone-point there is a saddle connection in $s_{\mathscr{F}} \in \mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$ with $|s| = |s_{\mathscr{F}}|$. In particular all horizontal saddle connections on $(\mathscr{F}_\tau, \omega_\tau)$ have integer length, if (S, α) covers $\mathbb{R}^2/\mathbb{Z}^2$. Moreover each rational point on a horizontal saddle connection of $\mathscr{F}_\tau$ represents an arithmetic surface (S, α), having a horizontal saddle connection s which connects the two cone points of (S, α). In fact as a point $(S, \alpha) \in s_{\mathscr{F}} \in \mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$ divides the saddle connection $s_{\mathscr{F}}$ into two pieces, say s^+ and s^-, and (S, α) contains horizontal saddle connections of length $|s^+|$ and of length $|s^-|$ connecting the two different cone points. There are two multiplicities $m^{\pm}_{s_{\mathscr{F}}}$ of saddle connections of length $s^{\pm}$ which are the same for all $(S, \alpha) \in s_{\mathscr{F}}$.

Theorem 3 (**Saddle connections**)**.** [S3] *Let $SC_h(\mathscr{F}_\tau) \subset \mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$ be the set of all horizontal saddle connections in $\mathscr{F}_\tau$, then taking the assumptions and notations of Theorem 2, we find for the asymptotic quadratic growth rate $c_{\pm}(\alpha)$ of saddle connections on $(S, \alpha) \in \mathscr{F}_\tau$ connecting the two different cone points of (S, α)*

$$c_{\pm}(\alpha) = \frac{2}{|\mathscr{O}_\alpha|} \sum_{s \in SC_h(\mathscr{F}_\tau)} \sum_{(Z,\nu) \in \mathscr{O}_\alpha(s)} \frac{m_s^+}{|s^+|^2}. \tag{5}$$

with $\mathscr{O}_\alpha(s) := \mathscr{O}_\alpha \cap s$ in the finite orbit case. For generic (S, α) we find

$$c_{\pm}(\alpha) = \frac{2\zeta(2)}{\mathrm{area}(\mathscr{F}_\tau)} \sum_{s \in SC_h(\mathscr{F}_\tau)} m_s^+ = \frac{2\zeta(2)}{\mathrm{area}(\mathscr{F}_\tau)} \sum_{p \in Z(\omega_\tau)} m_s^+ \cdot \widehat{o}_p, \tag{6}$$

where $\widehat{o}_p = o_p + 1$ and o_p is the order of the zero $p \in Z(\omega_\tau)$.

Remarks. One can break down these constants into smaller parts associated with the $\mathscr{U}$ orbit of a saddle connection $s_{\mathscr{F}} \in SC_h(\mathscr{F}_\tau, \omega_\tau)$, or $\mathrm{SL}_2(\mathbb{Z})$-orbits of cone points $p \in Z(\omega_\tau)$ in the generic case. The terms m^- and s^- do not appear in the expressions for the constants, because there is an isometric involution $\phi \in \mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau)$ of $\mathscr{F}_\tau$ which exchanges "+" and "−", causing the factor 2 in the expressions.

Let us also mention, that in case of our example the generic constant $c_\pm(\alpha)$ can be obtained by taking the limit over constants of any sequence of arithmetic surfaces $(S_i, \alpha_i) \in \mathscr{F}_\tau$, i.e.

$$\lim_{i\to\infty} c_\pm(\alpha_i) = c_\pm(gen)$$

while it is (in general) not true that

$$\sum_{(Z,\nu)\in\mathscr{O}_{\alpha_i}(s)} \frac{m_s^+}{|s^+|^2} \to \frac{2\zeta(2)}{\mathrm{area}(\mathscr{F}_\tau)} m_s^+, \quad \text{when } i \to \infty.$$

The reason for that has to do with the following observations:

- *For some saddle connections* $s_{\mathscr{F}} \in SC_h(\mathscr{F}_\tau, \omega_\tau)$ *there are sequences of surfaces* $\{(S_i, \alpha_i)\}_{i=1}^\infty$ *such that* $\mathscr{O}_{\alpha_i} \cap s_{\mathscr{F}} = \emptyset$ *for all* $i \in \mathbb{N}$.
- *The* $\mathrm{SL}_2(\mathbb{Z})$ *orbit of all surfaces contained in the set*

$$A := \mathscr{F}_\tau \setminus \mathrm{SL}_2(\mathbb{Z}) \cdot SC_h(\mathscr{F}_\tau) \subset \mathscr{F}_\tau$$

  avoids the set $\mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$.
  Note, that A *is obviously an* $\mathrm{SL}_2(\mathbb{Z})$*-invariant* G_δ*-set.*

d-symmetric differentials. The main goal of this paper is to carry out a completely explicit example illustrating Theorem 1 and evaluating the asymptotic constants in Theorem 2. To begin with, consider the following class of elliptic differentials:

We call (Y, τ) *d-symmetric*, if

- τ *has exactly* two *zeros of order* $d-1$
- $\mathbb{Z}/d\mathbb{Z} \subset \mathrm{Aut}(Y, \tau)$
- $\deg(\pi) = \int_X \pi^*(dx \wedge dy) = d$

with the natural projection $\pi : (Y, \tau) \to \mathbb{C}/\mathrm{Per}(\tau)$. Here $\mathrm{Aut}(Y, \tau)$ is the group of affine homeomorphisms of (Y, τ), preserving the holomorphic one form ω. The automorphism group is finite, if $g(Y) > 1$, since homeomorphisms of Y preserving τ are in fact holomorphic. We obtain d-symmetric differentials using the following:

Connected sum construction. Given an Abelian differential (X, ω) and a regular leaf $\mathscr{L} \in \mathscr{F}_\theta(X)$. Take $a \in \mathscr{L}$ and define the line segment

$$I := [0, \epsilon]e^{i\theta} + a \subset \mathscr{L}.$$

Then for $d \geq 2$ and a cycle $\sigma \in S_d$ we define the Abelian differential

$$\#^d_{I,\sigma}(X, \omega) \quad \text{or} \quad (\#^d_{I,\sigma} X, \#^d_{I,\sigma}\omega)$$

by slicing d named copies $X_1, ..., X_d$ of X along I and identify opposite sides of the slits according to the permutation σ. The notation $(\#^d_{I,\sigma}X, \#^d_{I,\sigma}\omega)$ is useful, if one needs to consider the differential $\#^d_{I,\sigma}\omega$, or the Riemann surface $\#^d_{I,\sigma}X$.

The differential $\#^d_{I,\sigma}\omega$ on $\#^d_{I,\sigma}X$ is uniquely defined by the property

$$\#^d_{I,\sigma}\omega|_{X_i} = \omega_i = \omega.$$

Note: we can rename the d copies of X such that the cycle σ becomes $\tau = (1, 2, 3, ..., d)$. In this case we simply write:

$$\#^d_I(X, \omega) = \#^d_{I,\tau}(X, \omega). \tag{7}$$

We apply this construction to a torus covering, by taking d copies of $\mathbb{T}^2$ and cut them along the projection of the line segment $I = I_v = [0, v] \subset \mathbb{C}$ to $\mathbb{T}^2$. For the first we assume the image of I_v on $\mathbb{T}^2$ is not a loop. Denote the result by

$$\#^d_I(\mathbb{T}^2, dz) \quad \text{or simply} \quad \#^d_I\mathbb{T}^2.$$

One extends this to all line segments $I_v = [0, v] \subset \mathbb{C}$, by choosing a sequence I_{v_i} of segments projecting injectively to $\mathbb{T}^2$, such that $I_{v_i} \to I_v$ when $i \to \infty$ and takes

$$\#^d_{I_v}(\mathbb{T}^2, dz) = \lim_{i\to\infty} \#^d_{I_{v_i}}(\mathbb{T}^2, dz)$$

on Teichmüller space.

The torus covers obtained by this connected sum construction have genus d and the translation structure has precisely two cone points of order d. For $d = 2$ we obtain the example from the introduction of part one of this paper and for $d = 1$ one obtains 2-marked tori studied in [S1]. By construction every $\#^d_I(\mathbb{T}^2, dz)$ has automorphism group

$$\mathbb{Z}/d\mathbb{Z} \cong \mathrm{Aut}(\#^d_I(\mathbb{T}^2, dz)) \subset \mathrm{Aff}^+(\#^d_I(\mathbb{T}^2, dz))$$

and is a d-fold torus cover. Therefore all $\#^d_I(\mathbb{T}^2, dz)$ are d-symmetric.

Denote the *set of isomorphy classes of d-symmetric coverings* of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by $\mathscr{F}^{sym}_d := \mathscr{F}^{sym}_d(d-1, d-1)$. Then

Theorem 4. *The modular fiber* $(\mathscr{F}^{sym}_d, \omega_d)$ *is given by*

$$(\mathscr{F}^{sym}_d, \omega_d) \cong (\mathbb{R}^2/d\mathbb{Z}^2 - \mathbb{Z}^2/d\mathbb{Z}^2, dz).$$

The integer lattice $\mathbb{Z}^2/d\mathbb{Z}^2 \subset \mathbb{T}^2_d := \mathbb{R}^2/d\mathbb{Z}^2$ *represents degenerated differentials (with one cone point) in the compactification of* $\mathscr{F}^{sym}_d$*.*

Every d-symmetric differential in $\mathscr{F}^{sym}_d$ *is represented by* $\#^d_I(\mathbb{T}^2, dz)$ *for an* $I \subset \mathbb{T}^2$*.*

There is an obvious, but interesting consequence:

Corollary 1. *Given natural numbers g and $n \geq 2$, there is a d-symmetric differential $(Y, \tau) \in \mathscr{F}_g^{sym}$, having the congruence group $\Gamma_1(n)$ as Veech group. In fact an (Y, τ) with this Veech group is given by*

$$[\#^d_{g/n}(\mathbb{T}^2, dz)] = [g/n + i0] \in \mathbb{C}/g(\mathbb{Z} \oplus \mathbb{Z}i).$$

Note that $\#^d_{g/n}(\mathbb{T}^2, dz)$ is degenerated, if and only if $g/n \in \mathbb{Z}$, or in other words $n|g$. Thus given $n \geq 2$, there are always infinitely many non-degenerated d-symmetric differentials with Veech group $\Gamma_1(n)$ as well as infinitely many degenerated ones with the same Veech group.

For d-symmetric differentials we evaluate formula 3 and find:

Theorem 5. *Let $(S, \alpha) \in \mathbb{T}^2_d$ be d-symmetric and $(S, \alpha) \notin \mathbb{Q}^2/d\mathbb{Z}^2$, i.e. has infinite $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{T}^2_d$. Then the asymptotic quadratic growth rate of periodic cylinders c_{cyl} on S is:*

$$c_{cyl}(\alpha) = 2 \sum_{p|d} \frac{\varphi(p)}{p^3}. \tag{8}$$

We evaluate formula 4 for d-symmetric differentials $(S, \alpha) \in \mathbb{Q}^2/d\mathbb{Z}^2 \subset \mathbb{T}^2_d$, the *torsion points* in $\mathscr{F}_d^{sym}$, as well. Finally we analyze special subsets of all saddle connections connecting the two cone points on $(X, \omega) \in \mathscr{F}_d^{sym}$. It turns out, that for given d and all $1 \leq m \leq d$ with $m|d$, there are a chains, each consisting of d/m parallel saddle connections which degenerate simultaneously, i.e. have the same length. We call such configuration of saddle connections *m-homologous*. Together all d/m chains belonging to a set of m-homologous saddle connections define a boundary in homology. We write down the asymptotic constants for m-homologous saddle connections for given $(X, \omega) \in \mathscr{F}_d^{sym}$.

Convention. For objects related to modular fibers we will use script fonts like $\mathscr{F}$, $\mathscr{C}$ and $\mathscr{L}$, while for (objects on) translation surfaces, which are not modular fibers, we use calligraphic fonts $\mathscr{C}$ and $\mathscr{L}$.

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2. Background

Geodesic segments on elliptic differentials. For any $\theta \in [0, 2\pi]$ the direction foliations $\mathscr{F}_\theta$ given by $\ker(\sin(\theta)dx + i\cos(\theta)dy)$ on $\mathbb{C}$ define foliations $\mathscr{F}_\theta(\mathbb{C}/\Lambda)$ on any torus $\mathbb{C}/\Lambda$. The *leaves* of $\mathscr{F}_\theta(\mathbb{C}/\Lambda)$ are straight lines and geodesics on $\mathbb{C}/\Lambda$ with respect to the metric $|dz|^2 = dx^2 + dy^2$. These objects pull back to any elliptic differential (X, ω), i.e. there are

direction foliations $\mathscr{F}_\theta(X,\omega)$ having integral curves which are straight lines with respect to the metric $|\omega|^2$.

Given a covering $\pi : X \to \mathbb{C}/\Lambda$ with $\omega = \pi^* dz$, then a leaf $\mathscr{L} \in \mathscr{F}_\theta(X,\omega)$ is compact if and only if $\pi(\mathscr{L}) \in \mathscr{F}_\theta(\mathbb{C}/\Lambda, dz)$ is compact. In particular, if (X,ω) is a cover of $(\mathbb{T}^2, dz) := (\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i, dz)$, leaves of $\mathscr{F}_\theta(X,\omega)$ are compact whenever $\tan(\theta) \in \mathbb{Q}$. A compact leaf which either has integral curve γ, or is represented by a homology cycle $[\gamma]$ is denoted by $\mathscr{L}_\gamma$. We call a leaf $\mathscr{L}$ *regular*, if it does not contain any zeros of ω.

Dynamics of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$. We collect facts on the $\mathrm{SL}_2(\mathbb{Z})$ operation on 2-*tori* $\mathbb{T}^2_{[0]} = (\mathbb{R}/\mathbb{Z}^2, dz, [0])$ (marked in $[0]$). Recall that marked tori are the modular fibers for moduli spaces of d-symmetric differentials.

It is clear that the affine group $\mathrm{Aff}^+(\mathbb{T}^2, dz)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ with the multiplication

$$(A,[a])\circ(B,[b]) = (A\cdot B, [b + Aa]) \quad \text{where } a,b\in\mathbb{R}^2 \quad \text{and } A,B\in\mathrm{SL}_2(\mathbb{Z}).$$

Clearly the short exact sequence of affine map becomes

$$0 \longrightarrow \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i \xrightarrow{\ i\ } \mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i \xrightarrow{\ \mathrm{D}\ } \mathrm{SL}_2(\mathbb{Z}) \longrightarrow 1 \tag{9}$$

As mentioned above, we mark one point on $\mathbb{T}^2$, say the origin $[0]$, to get rid of translations $\mathrm{Aut}(\mathbb{T}^2, dz) \cong \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$.

By $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz) \subset \mathrm{SL}_2(\mathbb{Z})$ we denote the stabilizer of the point $[x]\in\mathbb{T}^2$. The projection of $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz)$ to $\mathrm{PSL}_2(\mathbb{R})$ is the Veech group of the two marked torus $\mathbb{T}^2_{[x]} = (\mathbb{R}/\mathbb{Z}^2, dz, [0], [x])$ and we can regard $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz)$ as the Veech group of $(\mathbb{T}^2_{[x]}, dz)$ with distinguished marked points. If and only if $[x] = x + \mathbb{Z}^2$ is rational $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz)$ is a lattice in $\mathrm{SL}_2(\mathbb{Z})$, in fact:

Proposition 1. *Given $a,b,n\in\mathbb{N}_0$ with $\gcd(a,b,n)=1$ then the stabilizer $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz)$ of $[x] = [\frac{a}{n},\frac{b}{n}]$ is conjugated to*

$$\Gamma_1(n) := \left\{ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right) \mod n \right\} \subset \mathrm{SL}_2(\mathbb{Z}).$$

Moreover for $[x] = [\frac{1}{n}, 0]$: $\mathrm{SL}(\mathbb{T}^2_{[x]}, dz) = \Gamma_1(n)$.

This is easy to see, compare [S1] up to the fact that there we allow affine maps which interchange the marked points. The linear part of this bigger affine group is isomorphic to $\Gamma_1^{\pm}(n) := \{\pm A : A\in\Gamma_1(n)\}$.

The $\mathrm{SL}_2(\mathbb{Z})$-orbit of $[\frac{1}{n}, 0]\in\mathbb{T}^2$ is the set

$$\begin{aligned}\mathscr{O}_n = \left\{ \left[\frac{a}{n} + i\frac{b}{n}\right] \in \mathbb{T}^2 : \ a,b,n\in\mathbb{Z} \text{ with } \gcd(a,b,n) = 1 \right\} = \\ = \mathrm{SL}_2(\mathbb{Z})\cdot\left[\frac{1}{n}\right],\end{aligned} \tag{10}$$

in particular

$$[\Gamma_1(n) : \mathrm{SL}_2(\mathbb{Z})] = |\mathscr{O}_n| = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = \varphi(n)\psi(n).$$

The product is taken over all prime divisors p of n. We have also used the *Euler φ function* and the *Dedekind ψ function*:

$$\varphi(n) := n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad \psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right). \tag{11}$$

Torsion points on $\mathbb{T}^2_d$. A point on $\mathbb{T}^2_d$, contained in

$$\mathbb{T}^2_d[m] := \ker(\mathbb{T}^2_d \xrightarrow{m} \mathbb{T}^2_d) = \frac{d}{m}\mathbb{Z}^2/d(\mathbb{Z}^2).$$

for some $m \in \mathbb{N}$ is called *torsion* point. The homomorphism denoted by m is $z \mapsto m \cdot z$ for $z \in \mathbb{T}^2_d$. The *order* of a torsion point $z \in \mathbb{T}^2_d$ is the smallest $m \in \mathbb{N}$ such that $m \cdot z = 0 \in \mathbb{T}^2_d$. We denote the set of torsion points of order m on $\mathbb{T}^2_d$ by $\mathbb{T}^2_d(m)$. On the standard torus $\mathbb{T}^2 = \mathbb{T}^2_1$ we have by equation 10

$$\mathbb{T}^2(m) = \mathrm{SL}_2(\mathbb{Z}) \cdot [1/m] = \mathscr{O}_m. \tag{12}$$

Now the isomorphism

$$\mathbb{T}^2_d \xrightarrow{d^{-1}} \mathbb{T}^2, \quad z \mapsto d^{-1} \cdot z$$

is $\mathrm{SL}_2(\mathbb{Z})$ equivariant and identifies torsion points of order m.

Cusps and fundamental directions. Take a lattice surface (X, ω), then Veech group $\mathrm{SL}(X, \omega)$ acts on the set of periodic directions $\mathbb{P}^1(\mathrm{hol}(X, \omega))$ of (X, ω). A representative $v_i \in \mathbb{P}^1(\mathrm{hol}(X, \omega))$ for the orbit $\mathbb{P}^1(\mathrm{SL}(X, \omega) \cdot v_i)$ is called a *fundamental direction*. The set of fundamental directions is finite and in bijection to the cusps of $\mathrm{SL}(X, \omega)$.

3. Asymptotic formulas for branched covers of Veech surfaces

Now we develop a formalism for evaluating quadratic growth rates for branched covers of Veech surfaces. In case the covering is itself Veech, we derive the asymptotic formula based on the formula found by Gutkin and Judge [GJ] and independently by Vorobetz [Vrb]. In [S2] we use the Siegel-Veech formula to obtain Siegel-Veech constants for any dimensions of $\mathscr{F}$.

We start with the following data:

1. *a Veech-surface (X, ω) with Veech-group* $\mathrm{SL}(X, \omega)$
2. *a branched (translation) cover $p : (Y, \tau) \to (X, \omega)$ with lattice Veech group*
$$\mathrm{SL}(Y, \tau) \subseteq \mathrm{SL}(X, \omega).$$
3. *a space of covers $\mathscr{F}_\omega$ of (X, ω), such that* $\overline{\mathrm{SL}(X, \omega) \cdot (Y, \tau)} \subseteq \mathscr{F}_\omega$

Note: in general assumption 2 is a condition (which can always be reached by naming some objects on the cover). For arithmetic differentials (Y,τ) however, the (absolute) period lattice $\Lambda = \mathrm{Per}(\tau) \in \mathbb{C}$ gives the covering map $p : (Y,\tau) \to (\mathbb{C}/\Lambda, dz)$, implying $\mathrm{SL}(Y,\tau) \subset \mathrm{SL}(\mathbb{C}/\Lambda, dz)$.

For an arbitrary covering $p : (Y,\tau) \to (X,\omega)$ we can achieve

$$\mathrm{Aff}^+(Y,\tau) \cong \mathrm{SL}(Y,\tau)$$

by taking the reduced translation structure,

$$(Y_{red}, \tau_{red}) := (Y/\operatorname{Aut}(Y,\tau), \pi_*\tau)$$

where $\pi : Y \to Y_{red}$ is the natural projection.

We continue with assumptions;

4. *let* $v_0, ..., v_n \in \mathbb{P}^1(\mathrm{SL}(X,\omega))$ *be a* set of fundamental directions *on the base surface* (X,ω)*. Without restrictions we assume* v_0 *is horizontal.*
5. *furthermore let* $C_{i,1}, ..., C_{i,k_i}$ *are the maximal cylinders in direction* v_i *on* (X,ω)*.*

The following asymptotic formula has been derived by Gutkin and Judge [GJ], compare also [EMM].

Proposition 2. *Assume the stabilizer* $N \subset \mathrm{SL}_2(\mathbb{R})$ *of* $v \in \mathbb{R}^2$ *has nontrivial intersection with a lattice* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *and let* A *be a generator of* $N \cap \Gamma$*, then*

$$| \Gamma \cdot v \cap B(T) | \sim \mathrm{vol}(\mathbb{H}/\Gamma)^{-1} \frac{< Av^\perp, v >}{|v^\perp||v|^3} T^2, \tag{13}$$

where $v^\perp$ *is perpendicular to* v*.*

Without restrictions we take $v^\perp = \left[\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right]$ and $v = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right] \in \mathbb{R}^2$ and might think of it as a cylinder of height h and width w contained in the horizontal foliation of (X,ω). Now formula 13 reads

$$| \mathrm{SL}(X,\omega) \cdot v \ \cap \ B(T) | \sim \mathrm{vol}(\mathbb{H}/\Gamma)^{-1} \frac{< u_v v^\perp, v >}{h \cdot w^3} T^2 = \mathrm{vol}(\mathbb{H}/\Gamma)^{-1} \frac{l_v}{w^2} T^2, \tag{14}$$

with a linear Dehn twist given by

$$u_v = \left[\begin{smallmatrix} 1 & l_v \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SL}(X,\omega)$$

where $l_v \frac{h}{w} = p/q \in \mathbb{Q}$.

Now we relate the above formula for the base surface (X,ω) to the one for the cover (Y,τ). We already saw:

- *There is an operation of* $\mathrm{SL}(X,\omega)$ *on the set of coverings* $(Y,\tau) \to (X,\omega)$*. In particular there is a transitive operation on the orbit* $\mathrm{SL}(X,\omega) \cdot (Y,\tau)$ *in moduli space* $\mathscr{F}_\tau$*.*
- *a direction* v *on* (X,ω) *is completely periodic, if and only if it is completely periodic on* (Y,τ)

- *for a completely periodic direction v the stabilizer $N_v(Y,\tau) \subset \mathrm{SL}(Y,\tau)$ on (Y,τ) is a subgroup of the stabilizer $N_v(X,\omega) \subset \mathrm{SL}(X,\omega)$ and*

(15)
$$[N_v(X,\omega) : N_v(Y,\tau)] = \text{ length of the orbit } N_v(X,\omega)\cdot(Y,\tau) =: i_v(Y,X).$$

The following observation is fundamental for all calculations and formulas in this section.

Combinatorial description of cusps on (Y,ω)**.** Take the base Veech surface (X,ω) and choose a fundamental direction v, again we *assume v is horizontal*. The fundamental directions $v_0,...,v_n$ of a Veech surface (X,ω) represent the cusps of $\mathbb{H}/\mathrm{SL}(X,\omega)$.

Proposition 3. *The number of cusps of (Y,ν) above a given cusp v on (X,ω) are in one-to-one correspondence to the u_v orbits on the set $\mathrm{SL}(X,\omega)/\mathrm{SL}(Y,\nu)$.*

Proof. For every direction $w \in \mathbb{P}^1(\mathrm{SL}(X,\omega)\cdot v)$ there is (at least one) surface

$$(S,\alpha) = A\cdot(Y,\nu) \in \mathrm{SL}(X,\omega)\cdot(Y,\nu)$$

for which $A\cdot w$ is horizontal. If there are two maps $A,B \in \mathrm{SL}(X,\omega)$ making the direction w horizontal, then $A\circ B^{-1} \in N_v(X,\omega)$. Thus the number of all $A\cdot(Y,\nu) \in \mathrm{SL}(X,\omega)\cdot(Y,\nu)$ such that $A\cdot w$ is horizontal equals $i_v(S,X)$, which is the length of the u_v orbit of (S,α). This proves the claim. □

We define the *relative width of a cusp* represented by a direction w over v on (Y,ν) as the length of the u_v (the stabilizer of v in $\mathrm{SL}(X,\omega)$) orbit

$$\{u_v^n\cdot(Y,\nu):\ n\in\mathbb{Z}\} \subset \mathrm{SL}(X,\omega)\cdot(Y,\nu).$$

Note: for arithmetic surfaces (= torus covers) the relative cusp width is the same as the usual cusp width. Also for arithmetic surfaces the above characterization of cusps was discovered by A. Zorich and published in [HL].

Together with equation 14 we have:

Proposition 4. *Under the above assumptions,let $v \in \mathbb{C}$ be a periodic direction on (X,ω) and (S,α) a differential in the orbit $\mathscr{O}_\tau := \mathrm{SL}(X,\omega)\cdot(Y,\tau)$, then*

$$(16)\quad |\ \mathrm{SL}(S,\alpha)\cdot v \cap B(T)\ | \sim \mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))^{-1}\frac{l_v\cdot i_v(S,X)}{[\mathrm{SL}(X,\omega):\mathrm{SL}(Y,\tau)]}\frac{1}{|v_i|^2}$$

Again assume v is a periodic direction on (X,ω), pick a periodic cylinder $c_X \subset \mathscr{F}_v(X,\omega)$, (or saddle connection $s_X \in \mathscr{F}_v(X,\omega)$). Take a pre-image c_S (or s_S) of c_X (s_X respectively) on a differential $(S,\alpha) \in \mathscr{O}_\tau$, then by 16 we have the asymptotic quadratic growth rate

$$(17)\qquad c_{cyl}(c_S) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{l_v\cdot i_v(S,X)}{[\mathrm{SL}(X,\omega):\mathrm{SL}(Y,\tau)]}\frac{1}{|w_{c_S}|^2}.$$

using $\mathscr{U}_\alpha := N_v(X,\omega)\cdot(S,\alpha)$ and $|\mathscr{O}_\tau| = [\mathrm{SL}(X,\omega) : \mathrm{SL}(Y,\tau)]$ we get

$$c_{cyl}(c_S) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{l_v}{|\mathscr{O}_\tau|}\sum_{(S,\alpha)\in\mathscr{U}_\alpha}\frac{1}{w^2} \tag{18}$$

Now for any fundamental direction w of (Y,τ) such that $w \in \mathrm{SL}(X,\omega)\cdot v$ there is a differential $(S,\alpha) = A\cdot(Y,\tau) \in \mathscr{O}_\tau$ such that $v = A\cdot w$. Thus we can apply equation 18 to all cylinders in a cylinder decomposition in direction w on (Y,τ) by looking at the cylinder decomposition $\mathscr{C}_{\alpha,1},\ldots,\mathscr{C}_{\alpha,n_\alpha}$ in direction $v = Aw$ on $(S,\alpha) = A\cdot(Y,\tau)$. Denote the width of the cylinder $\mathscr{C}_{\alpha,i}$ by $w_{\alpha,i}$, then the *asymptotic quadratic growth rate of all cylinders on* (Y,τ) *having direction contained in the set* $\mathrm{SL}(X,\omega)\cdot v$ is given by:

$$c_{cyl,\tau}(v) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{l_v}{|\mathscr{O}_\tau|}\sum_{(S,\alpha)\in\mathscr{O}_\tau}\sum_{i=1}^{n_\alpha}\frac{1}{w_{\alpha,i}^2}. \tag{19}$$

Siegel-Veech constants – finite orbit case. The Siegel-Veech constant for cylinders (or saddle connections) on any $(S,\alpha)\in\mathscr{O}_\tau$ is now obtained by taking the sum over all cylinders a fundamental directions $v_1,\ldots,v_n$ on (X,ω):

$$c_{cyl,\tau} = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{1}{|\mathscr{O}_\tau|}\sum_{(S,\alpha)\in\mathscr{O}_\tau}\sum_{k=1}^{n} l_{v_k}\sum_{i=1}^{n_{\alpha,k}}\frac{1}{w_{\alpha,k,i}^2} \tag{20}$$

Note that away from the *hyperbolic volume, an index and length of cylinders in a particular direction* one needs to know only the *set of fundamental directions on the base* (X,ω) and the number l_{v_i} associated to each fundamental direction.

Basic examples – arithmetic surfaces. Arithmetic surfaces (Y,τ) possess a natural map to a torus $\mathbb{C}/\Lambda$. Taking an $\mathrm{SL}_2(\mathbb{R})$ deformation of (Y,τ) we might assume $\Lambda = \mathbb{Z}\oplus\mathbb{Z}i$, the deformation does not change quadratic growth rates. Now on $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ there is only one fundamental direction, we can assume this direction is horizontal and $l_{v_i} = 1$. We write down the statements for several sub-types of saddle connections later, there are many interesting cases labeled by various properties to consider.

Note that for arithmetic surfaces the formula is completely general. We apply it in this paper after using the cylinder decomposition of the moduli space fiber to simplify and organize information.

Basic examples – non arithmetic Veech surfaces. Non arithmetic Veech surfaces cannot have a direction with one single closed cylinder. Thus the best situation is (X,ω) has

- *only* one fundamental direction v *with, say* k *cylinders and*
- *the moduli* $w_1/h_1 = \ldots = w_k/h_k =: m$ *of the cylinders are all the same*

here w_i is the width and h_i the height of the i-th cylinder. It follows $l_v = m$ and the the Siegel-Veech constant for cylinders is

$$c_{cyl,\omega} = \frac{\pi \cdot l_v}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))} \sum_{i=1}^{k} \frac{1}{w_i^2} = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))} \sum_{i=1}^{k} \frac{1}{h_i w_i}. \tag{21}$$

A new paper by Eskin, Marklof and Morris [EMM] contains a discussion of the Veech surfaces X_n obtained by linearization of the billiard in the triangle with angles $(\pi/n, \pi/n, (n-2)\pi/n)$. A result of Veech [V3] shows X_n has only one fundamental direction v and the cylinder moduli are all the same. Thus formula 21 applies (compare [EMM] pgs 26-28). In fact one can simplify using explicit expressions for the w_i and h_i, see [V3, EMM].

Both classes of examples are prototypes to consider branched covers above them. For a very particular case, 2-fold covers of X_n, this has been done in [EMM] to get information on some non Veech triangular billiards.

4. Counting using the horizontal foliation of the modular fiber.

One can organize the asymptotic formula 20 using modular fibers. Assume (X,ω) is Veech with lattice group $\mathrm{SL}(X,\omega)$ and $\pi : (Y,\tau) \to (X,\omega)$ is a (branched) covering. Define

$$\mathscr{E}_\tau := \overline{\mathrm{SL}_2(\mathbb{R}) \cdot (Y,\tau)} \subset \mathscr{H}_g \text{ and } \mathscr{F}_\tau := \overline{\mathrm{SL}(X,\omega) \cdot (Y,\tau)}.$$

The *closure* of these two spaces is taken within the set of Abelian differentials having the same zero order pattern as τ. There is another closure, denoted by $\mathscr{F}_\tau^c$ ($\mathscr{E}_\tau^c$ respectively), coming from the compactification of the fibers

$$A \cdot (\mathscr{F}_\tau, \omega_\tau), \quad A \in \mathrm{SL}_2(\mathbb{R})$$

viewed as Abelian differentials. The set $\mathscr{F}_\tau^c - \mathscr{F}_\tau$ parameterizes *degenerated differentials*, but eventually *not* up to isomorphy. In our definition of $\mathscr{E}_\tau = \mathscr{E}_\tau(o_1, ..., o_n)$ we distinguish cone points, which makes $\mathscr{F}_\tau$ into an Abelian differential, if $\dim(\mathscr{F}_\tau) = 2$ (see [S2]). In case (Y,τ) is Veech itself

$$\mathscr{E}_\tau^c = \mathscr{E}_\tau \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}(Y,\tau) \subset \Omega\mathscr{M}_g \tag{22}$$

and by taking base points with respect to $\pi : \Omega\mathscr{M}_g \to \mathscr{M}_g$ we obtain a Teichmüller curve in $\mathscr{M}_g$.

In this paper we are mainly interested in cases, when the dimension of $\mathscr{F}_\tau$ equals 0 or 2. Since we understand the 0-dimensional case already, we develop the counting theory for 2-dimensional fibers.

Proof of Theorem 1. For every $(S,\alpha)\in\mathscr{F}_\tau$ there is a map

$$\pi:(S,\alpha)\to(X,\omega)$$

which for arithmetic S becomes

$$\pi:(S,\alpha)\to(\mathbb{C}/\operatorname{Per}(\alpha),dz).$$

Now choose a symplectic base of the relative homology $H_1(Y,Z(\alpha);\mathbb{Z})$, say $(\gamma_1,...,\gamma_{2g},\gamma_{2g+1},...,\gamma_{2g+n})$. Given an ordering $z_0,...,z_n$ of the zeros $z_i\in Z(\alpha)$ the *relative cycles* $(\gamma_{2g+1},...,\gamma_{2g+n})$ can be chosen to have the property

$$\partial\gamma_{2g+i}=z_0-z_i.$$

To use the given symplectic base for other surfaces, we suppose every differential (S,α) comes with a marking $\phi_S:(Y,Z(\tau))\to(S,Z(\alpha))$, i.e. an orientation preserving homeomorphism mapping $Z(\tau)$ bijectively onto $Z(\alpha)$. This allows to identify $H_1(Y,Z(\tau);\mathbb{Z})$ with $H_1(S,Z(\alpha);\mathbb{Z})$. Using the (relative) homology base, one defines local coordinates of the connected component of the stratum $\mathscr{H}_1(o_0,...,o_n)$ containing (S,α)

$$(S,\alpha)\mapsto\left(\int_{\gamma_1}\alpha,...,\int_{\gamma_{2g}}\alpha,\int_{\gamma_{2g+1}}\alpha,...,\int_{\gamma_{2g+n}}\alpha\right)\subset\mathbb{C}^{2g+n}, \tag{23}$$

giving the stratum a complex manifold structure. The $\mathrm{SL}_2(\mathbb{R})$ action on differentials

$$A\cdot\alpha:=\left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}\mathrm{Im}\,\alpha\\ \mathrm{Re}\,\alpha\end{smallmatrix}\right)$$

obviously commutes with integration

$$\int_\gamma A\cdot\alpha=A\cdot\int_\gamma\alpha, \tag{24}$$

giving the standard real linear action of $\mathrm{SL}_2(\mathbb{R})$ on coordinates.

If (S,α) is arithmetic, the absolute periods $\gamma\in H_1(Y;\mathbb{Z})$ generate a lattice

$$\operatorname{Per}(\alpha):=\left\{\int_\gamma\alpha:\gamma\in H_1(Y,\mathbb{Z})\right\}\subset\mathbb{R}^2$$

and therefore a map of differentials

$$(S,\alpha)\to(\mathbb{C}/\operatorname{Per}(\alpha),dz).$$

By definition of the $\mathrm{SL}_2(\mathbb{R})$-action on differentials

$$\operatorname{Per}(\mathrm{SL}(\mathbb{C}/\operatorname{Per}(\alpha),dz)\cdot\alpha)=\operatorname{Per}(\alpha),$$

which implies that $\mathrm{SL}(\mathbb{C}/\operatorname{Per}(\alpha),dz)$ acts on the set of differentials $(X,\omega)\in\overline{\mathrm{SL}_2(\mathbb{R})\cdot(Y,\tau)}$ with absolute period lattice $\operatorname{Per}(\omega)=\operatorname{Per}(\alpha)$. For simplicity we assume from now on $\operatorname{Per}(\alpha)\cong\mathbb{Z}^2$ and thus all differentials contained in the modular fiber

$$\mathscr{F}_\tau:=\overline{\mathrm{SL}_2(\mathbb{Z})\cdot(Y,\tau)}$$

have absolute period lattice $\mathbb{Z}^2$, by continuity of local coordinates (23). The difference of any two *relative periods* $\gamma_1, \gamma_2 \in H_1(Y, Z(\tau); \mathbb{Z})$ with

$$\partial\gamma_1 = z_0 - z_i = \partial\gamma_2$$

is an absolute period, hence there is a well-defined map

$$\pi : \begin{cases} \mathscr{F}_\tau \longrightarrow & \mathbb{T}^2 \times \cdots \times \mathbb{T}^2 \\ (S, \alpha) \longmapsto & (\int_{z_0}^{z_1} \alpha \bmod \mathbb{Z}^2, ..., \int_{z_0}^{z_n} \alpha \bmod \mathbb{Z}^2) \end{cases} \tag{25}$$

whenever γ_i is a relative cycle with $\partial\gamma_i = z_0 - z_i$. By definition of local coordinates this map is holomorphic and by construction of $\mathscr{F}_\tau$ $\mathrm{SL}_2(\mathbb{Z})$-equivariant. In fact, since the absolute period lattice is the same for all surfaces parameterized by $\mathscr{F}_\tau$, the absolute period coordinates are locally constant on $\mathscr{F}_\tau$ and thus π provides local coordinates of $\mathscr{F}_\tau$. This implies together with the $\mathrm{SL}_2(\mathbb{Z})$ equivariance that $\pi : \mathscr{F}_\tau \to \pi(\mathscr{F}_\tau) \subset (\mathbb{T}^2)^n$ is a covering with image a union of tori of dimension equal of $\dim(\mathscr{F}_\tau)$.

By assumption we are interested in differentials (Y, τ) having exactly 2 cone points covering $\mathbb{T}^2$ in a way that the relative location of the cone points is torsion. This means we obtain a surjective cover

$$\pi : \mathscr{F}_\tau \to \mathbb{T}^2,$$

since the π image of the relative period coordinate is not closed under $\mathrm{SL}_2(\mathbb{Z})$. Thus $\mathscr{F}_\tau$ is an arithmetic surface, by construction it is square-tiled.

Now we can pull back dz to get a holomorphic 1-form $\omega_\tau = \pi^* dz$ on $\mathscr{F}_\tau$, thus obtaining an elliptic differential

$$(\mathscr{F}_\tau, \omega_\tau) \quad \text{with} \quad \mathrm{Per}(\omega_\tau) \subset \mathbb{Z}^2,$$

which has by definition $\mathrm{D}\,\mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau) = \mathrm{SL}_2(\mathbb{Z})$. By means of its definition as a stabilizer group one would not call $\mathrm{SL}_2(\mathbb{Z})$ the Veech group of $(\mathscr{F}_\tau, \omega_\tau)$, if $\mathscr{F}_\tau$ is not connected. On the other hand each deformation of a surface $(S, \alpha) \in \mathscr{F}_\tau$ along a path $t \mapsto \gamma(t) \in \mathrm{SL}_2(\mathbb{R})$, such that $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A \in \mathrm{SL}_2(\mathbb{Z})$ represents a homotopy class

$$[\gamma] \in \pi_1(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{SL}_2(\mathbb{Z}),$$

which in turn defines an affine homeomorphism

$$\phi_\gamma : \begin{cases} \mathscr{F}_\tau \to \mathscr{F}_\tau \\ (S, \alpha) \mapsto A \cdot (S, \alpha). \end{cases} \tag{26}$$

That the map is affine with derivative $A \in \mathrm{SL}_2(\mathbb{Z})$, follows immediately from the previous definition of the translation structure on $\mathscr{F}_\tau$ and equation (24). Thus we get isomorphisms

$$\mathrm{D}\,\mathrm{Aff}^+(\mathscr{F}_\tau, \omega_\tau) \cong \mathrm{SL}_2(\mathbb{Z}) \cong \pi_1(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})).$$

Cylinder decomposition of $(\mathscr{F}_\tau, \omega_\tau)$. As an cover of $\mathbb{T}^2$ the differential $(\mathscr{F}_\tau, \omega_\tau)$ is completely periodic in each *rational* direction, in particular in the horizontal direction. Thus we only need to show, that every $(S, \alpha) \in$

$\mathscr{F}_\tau$ admits a completely periodic horizontal foliation. But this is obvious, because

$$\mathrm{Per}(\alpha) = \mathbb{Z}^2 \supset \mathrm{Per}(\omega_\tau).$$

The above generalizes in an obvious way to any deformed fiber $A \cdot (\mathscr{F}_\tau, \omega_\tau)$, $[A] \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Moreover one can extend this result to 2-dimensional, $\mathrm{SL}(X,\omega)$-invariant subspaces in X^n, for any (reduced) Veech surface (X,ω) (see [HST]).

Stability of cylinder-decompositions. The case we are interested in is (Y,τ) is a non-arithmetic cover $\pi : Y \to \mathbb{T}^2$ with exactly 2 cone points p_0 and p_1, we assume $\pi(p_0) = [0] \in \mathbb{T}^2$. This implies the modular fiber $\mathscr{F}_\tau := \overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (Y,\tau)}$ has dimension 2. A relative period, say $\gamma_1 \in H_1(Y, \{p_0,p_1\}; \mathbb{Z})$, with $\partial\gamma_1 = p_1 - p_0$, provides local coordinates

$$(S,\alpha) \mapsto z(p_1) = \int_{\gamma_1} \alpha$$

on $\mathscr{F}_\tau$. If we choose any absolute cycle $[\gamma] \in H_1(Y;\mathbb{Z}) \subset H_1(Y,\{p_0,p_1\};\mathbb{Z})$, the function

$$\mathscr{F}_\tau \ni (S,\alpha) \mapsto \int_\gamma |\alpha| \in \mathbb{N} \tag{27}$$

is continuous, hence constant on connected components of $\mathscr{F}_\tau$. In particular, if γ is represented by a *regular leaf* of $\mathscr{L} \in \mathscr{F}_\theta(S,\alpha)$, i.e.

$$\int_{\mathscr{L}} |\alpha| = |\int_{\mathscr{L}} \alpha| \in \mathbb{N} \tag{28}$$

there is a neighborhood of $(S,\alpha) \in \mathscr{F}_\tau$ where γ is represented by a regular leaf of the same length as $\mathscr{L}$. Indeed regular leaves $\mathscr{L} \in \mathscr{F}_\theta(S,\alpha)$ have positive distances from the cone point p_1, and all these distances are continuous functions depending on (the coordinates of) $(S,\alpha) \in \mathscr{F}_\tau$. Since the length function 27 is constant on each connected component of $\mathscr{F}_\tau$ for curves γ such that $[\gamma] \in H_1(Y;\mathbb{Z})$, the only possible thing which can happen under deformation is that γ looses the property of being realized by a regular leaf (if it is at all). This in turn happens only if the cone point p_1 crosses the leaf $\mathscr{L}$ realizing γ. If we now fix a periodic direction $\theta \in S^1$ on $(S,\alpha) \in \mathscr{F}_\tau$, the same direction is is periodic on $\mathscr{F}_\tau$. Now assume $(S_t,\alpha_t) \in \mathscr{L} \in \mathscr{F}_\theta(\mathscr{F}_\tau,\omega_\tau)$, $\mathscr{L}$ regular and $\{(S_t,\alpha_t) :\ t \in \mathbb{R}\} = \mathscr{L}$. Then by the above all regular leaves of any (S_t,α_t) in direction $\theta \in S^1$ stay regular since we move the cone point p_1 in direction θ. This completes the proof. □

Elliptic differentials. As in the previous proof, we assume $\mathscr{F}_\tau = \mathscr{F}_\tau(o_1,o_2)$ is a fiber space of *elliptic differentials* (S,α) with two cone points of ord er o_i and period lattice $\mathrm{Per}(\alpha) = \mathbb{Z}^2$.

Each horizontal cylinder $\mathscr{C}_i$ of $\mathscr{F}_\tau$ is bounded by a union of saddle connections $\partial^{top}\mathscr{C}_i$ parametrizing elliptic differentials $(S,\alpha) \in \mathscr{F}_\tau \cap \partial^{top}\mathscr{C}_i$ where the two cone points $z_0, z_1 \in Z(\alpha)$ are connected by (at least one) horizontal saddle connection, say $s \in \mathscr{F}_h(S,\alpha)$. Thus deforming (S,α) along

$\partial^{top}\mathscr{C}_i$ is nothing but collapsing the two cone points of (S,α) into one. This means points on $\partial^{top}\mathscr{C}_i$ contain (degenerated) elliptic differentials of type $\mathscr{E}(o)$ where $|o_1 - o_2| \leq o \leq o_1 + o_2$.
Remark. We will see, that these degenerated surfaces are not necessary cone points of $(\mathscr{F}_\tau, \omega_\tau)$.

Now the cylinder decomposition of $(S,\alpha) \in \mathscr{F}_\tau$ is not affected when deforming (S,α) inside $\mathscr{C}_i$, however the height of some horizontal cylinders on (S,α) disappears when deforming into $\partial^{top}\mathscr{C}_i$. Hence there are some horizontal cylinders on (S,α) which degenerate to saddle connections when deforming (S,α) into $\partial^{top}\mathscr{C}_i$. The only thing we use is that the (width of the) cylinders on (S,α) which degenerate do not depend on point where we enter $\partial^{top}\mathscr{C}_i$ through $\mathscr{C}_i$.

This allows to simplify counting formula (20) above using the cylinder decomposition of $\mathscr{F}_\tau(o_1,o_2)$ *and* the $\mathrm{SL}_2(\mathbb{Z})$-orbit $\mathscr{O}_\alpha$ of $(S,\alpha) \in \mathscr{F}_\tau(o_1,o_2)$:

$$c_{cyl}(\alpha) = \frac{1}{|\mathscr{O}_\alpha|} \sum_{i=1}^{n_{\mathscr{F}}} \left[\sum_{k=1}^{n_i} \frac{|\mathscr{O}_\alpha \cap \mathscr{C}_i|}{w_{i,k}^2} + \sum_{k=1}^{m_i} \frac{|\mathscr{O}_\alpha \cap \partial^{top}\mathscr{C}_i|}{w_{i,k}^2} \right]. \tag{29}$$

Calculating quadratic constants using sequences. Choose a nonperiodic direction $\theta \in S^1$ on $(S,\alpha) \in \mathscr{F}_\tau$, or equivalently on $\mathscr{F}_\tau$, take the unipotent subgroup $u_\theta(t) \subset \mathrm{SL}_2(\mathbb{R})$ and look at the orbit

$$\mathscr{O}_{\theta,\alpha}(T) := \{u_\theta(t) \cdot (S,\alpha) : \ t \in (0,T)\}$$

Using the Siegel-Veech formula and ergodicity of the u_θ action on $\overline{\cup_{T>0}\mathscr{O}_{\theta,\alpha}(T)}$, one can calculate all kinds of Siegel-Veech constants. Given the shape of the Siegel-Veech formula one can try to use $\mathscr{O}_{\theta,\alpha}(T)$, to approximate Siegel-Veech constants for (S,α) (however this can be done). One expects to be close to the real Siegel-Veech constant for (S,α), when T is large enough. However the orbit we have chosen hits the modular fiber $\mathscr{F}_\tau$ only once. Thus one needs to consider unipotent subgroups which orbit returns to $\mathscr{F}_\tau$ often, i.e. the unipotent subgroups generated by parabolic elements fixing a periodic direction of $\mathscr{F}_\tau$. Depending on the chosen point $(S,\alpha) \in \mathscr{F}_\tau$ we see a finite or infinite intersection

$$\overline{\cup_{T>0}\mathscr{O}_{\theta,\alpha}(T)} \cap \mathscr{F}_\tau,$$

in the infinite case we see (a union of) leaves

$$\mathscr{L} = \overline{\cup_{T>0}\mathscr{O}_{\theta,\alpha}(T) \cap \mathscr{F}_\tau} \subset \mathscr{F}_\theta(\mathscr{F}_\tau, \omega_\tau).$$

In the finite orbit case we like to ask, if a sequence of finite orbit surfaces $(S_i,\alpha_i) \in \mathscr{F}_\tau$ with $\lim_{i\to\infty} |\mathscr{O}_{\alpha_i}| = \infty$ admits the continuity property

$$\lim_{i\to\infty} c_*(S_i,\alpha_i) = c_{*,gen}(S,\alpha)$$

for any type of Siegel-Veech constant c_*. Here $c_{*,gen}$ is the Siegel-Veech constant for generic surfaces in $\overline{\cup_i \mathscr{O}_{\alpha_i}}$. Because $\mathscr{F}_\tau$ is an arithmetic surface

itself, we expect

$$\overline{\cup_i \mathscr{O}_{\alpha_i}} = \mathscr{F}_\tau.$$

To avoid a general discussion, *assume* for the moment the orbits $\mathscr{O}_{\alpha_i}$ become more and more equally distributed in $\mathscr{F}_\tau$ with respect to Lebesque measure. This is known to be true for $(S, \alpha) \in \mathscr{F}_\tau$ with infinite $\mathrm{SL}_2(\mathbb{Z})$ orbit, in fact [EMM, EMS] implies that $\mathscr{O}_\alpha$ is equally distributed in $\mathscr{F}_\tau$ with respect to Lebesque measure. Using either series of finite orbits (with equal distribution assumption), or infinite orbits, we obtain the asymptotic constant for *generic differentials* as a limit from formula (29)

$$c_{cyl}(\alpha) = \sum_{i=1}^{n_{\mathscr{F}}} \frac{\operatorname{area}(\mathscr{C}_i)}{\operatorname{area}(\mathscr{F}_\tau)} \sum_{k=1}^{n_i} \frac{1}{w_{i,k}^2}, \tag{30}$$

because by equal distribution in the limit

$$\frac{|\mathscr{O}_{\alpha_i} \cap\ \mathscr{C}_j|}{|\mathscr{O}_{\alpha_i}|} \overset{i\to\infty}{\longrightarrow} \frac{\operatorname{area}(\mathscr{C}_j)}{\operatorname{area}(\mathscr{F}_\tau)} \quad \text{and} \quad \frac{|\mathscr{O}_{\alpha_i} \cap\ \partial^{top}\mathscr{C}_j|}{|\mathscr{O}_{\alpha_i}|} \overset{i\to\infty}{\longrightarrow} \frac{\operatorname{area}(\partial^{top}\mathscr{C}_j)}{\operatorname{area}(\mathscr{F}_\tau)} = 0. \tag{31}$$

Without giving a completely general proof (for equal-destribution or convergence) at this place, we verify convergence of asymptotic constants in case of d-symmetric differentials (in most cases).

Counting saddle connections. The asymptotic formula for saddle connections is the same as the one for cylinders as long as $(S, \alpha) \in \mathscr{F}_\tau$ is arithmetic (or in general: Veech). One just needs to replace the width w_i of cylinders in the horizontal direction by the length of the bounding saddle connections s_j in formula 29. Things become different if we take $(S, \alpha) \in \mathscr{F}_\tau$ and ask for the quadratic growth rate of saddle connections which connect the two (different) singular points $z_0, z_1 \in Z(\alpha)$ of (S, α). If a line segment connecting z_0 with z_1 is horizontal, (S, α) is located on a saddle connection in $\mathscr{F}_h(\mathscr{F}_\tau)$, because we can degenerate (S, α) by moving z_1 into z_o along a saddle connection $s_{\mathscr{F}} \in \mathscr{F}_h(\mathscr{F}_\tau)$. Now the saddle connections on (S, α) which are killed by this deformation are exactly of the length of the path necessary to degenerate (S, α) along $s_{\mathscr{F}} \in \mathscr{F}_h(\mathscr{F}_\tau)$, or in other words the length of these saddle connections is the distance of $(S, \alpha) \in s_{\mathscr{F}} \subset \mathscr{F}_\tau$ to the endpoint of $s_{\mathscr{F}}$.

There are two possible directions (left or right) to move (S, α) into a cone point of $\mathscr{F}_\tau$) along the saddle connection $s_{\mathscr{F}}$. We denote the distance of $(S, \alpha) \in s_{\mathscr{F}}$ to the left, or right endpoint of $s_{\mathscr{F}}$ by s_α^-, s_α^+ respectively. On each $(S, \alpha) \in s_{\mathscr{F}}$ there are $m_s^\pm$ saddle connections of length $s_\alpha^\pm$ which disappear when moving (S, α) into the endpoints on $s_{\mathscr{F}}$. Using this and formula 29 we find the quadratic asymptotic constant $c_\pm(Y, \tau)$ for the set of saddle connections $SC^\pm$ on $(Y, \tau) \in \mathscr{F}_\tau$ connecting two (different) cone points z_0

and z_1:

$$c_{\pm}(\tau) = \frac{1}{|\mathscr{O}_\tau|} \sum_{s \in SC_h(\mathscr{F}_\tau)} \sum_{\alpha \in \mathscr{O}_\tau(s)} \left[\frac{m_s^-}{(s_\alpha^-)^2} + \frac{m_s^+}{(s_\alpha^+)^2} \right] = \tag{32}$$

$$= \frac{2}{|\mathscr{O}_\tau|} \sum_{s \in SC_h(\mathscr{F}_\tau)} \sum_{\alpha \in \mathscr{O}_\tau(s)} \frac{m_s^\pm}{(s_\alpha^\pm)^2}.$$

with $\mathscr{O}_\tau(s) := \mathscr{O}_\tau \cap s$. The last identity follows from the existence of an involution $\phi \in \mathrm{Aff}^+(\mathscr{F}_\tau)$ with $\mathrm{D}\,\phi = -\,\mathrm{id}$ acting on $SC_h(\mathscr{F}_\tau, \omega_\tau)$.

Remarkably one can also obtain Siegel-Veech constants $c_\pm$ for generic surfaces in $\mathscr{F}_\tau$ using sequences of arithmetic differentials (S_i, α_i). At hand of the modular fibers for d-symmetric differentials one sees, that the smallest pieces into which we can decompose the generic constant

$$c_{\pm,gen} = \sum_{\mathscr{O}_{deg}} c_\pm(\mathscr{O}_{deg})$$

such that

$$\lim_{i \to \infty} c_\pm(S_i, \alpha_i) = c_\pm(\mathscr{O}_{deg})$$

are the constants associated to $\mathrm{SL}_2(\mathbb{Z})$ orbits of degenerated differentials in $\mathscr{F}_\tau^c$.

We show convergence of constants for 2-symmetric differentials on pgs. 37 – 38. Comparing our method, i.e. using sequences, with the direct calculation using the Siegel-Veech formula we observe that only the number of intersections

$$\sum_{p \in \mathscr{O}_q} |\mathscr{O}_n \cap B_p(\varepsilon)|, \quad \frac{1}{q} \in Z(\omega_\tau) \subset \mathscr{F}_d^{sym,c}$$

with a small disc $B_p(\varepsilon)$ of radius ε, centered at a cone point p plays a role. To get the right generic constant in the limit $n \to \infty$ $\mathscr{O}_n$ must become more and more equally distributed in $\mathscr{F}_\tau$ since only then:

$$\frac{1}{|\mathscr{O}_n|} \sum_{p \in \mathscr{O}_q} |\mathscr{O}_n \cap B_p(\varepsilon)| \overset{n\to\infty}{\longrightarrow} \frac{|\mathscr{O}_q|}{\mathrm{area}(\mathscr{F}_\tau)} \pi \varepsilon^2.$$

Note, that in this case it is not necessary to consider distances of a particular point or surface $(S, \alpha) \in \mathscr{O}_n \cap B_q(\varepsilon)$ to the cone point q like in the previous formulas.

It is common, see [EMZ] and [EMS], to divide the generic asymptotic constant into constants by prescribing *orders of cone points* $o = o(z)$, $z \in Z(\omega_\tau)$.

$$c_{gen}^\pm = \sum_{o = o(z),\ z \in Z(\omega_\tau)} c_{gen,o}^\pm \tag{33}$$

with

$$c^{\pm}_{gen,o} = 2\frac{\zeta(2)}{\operatorname{area}(\mathscr{F}_\tau)}|\{z \in Z(\omega_\tau) : o(z) = o\}| \cdot o \cdot m_o. \tag{34}$$

Note that sets with prescribed orders of cone points are $\mathrm{SL}_2(\mathbb{Z})$-invariant and that our previous formulas represent a decomposition of constants which goes beyond fixing cone-point orders.

Other asymptotic constants. Assume $\mathscr{F}_\tau := \mathscr{F}_\tau(o_1, o_2)$ is a fiber of a 2-dimensional space of elliptic differentials having two cone points of order o_1 and o_2. Let $\mathscr{C}_1, ..., \mathscr{C}_n$ be the decompositon of the horizontal foliation $\mathscr{F}_h(\mathscr{F}_\tau, \omega_\tau)$ into maximal, open-cylinders. We might take an open cylinder $\mathscr{C}_{i,(a,b)} \subset \mathscr{C}_i$. If $t_v \in (0, h_i)$ is a transversal coordinate of $\mathscr{C}_i$ (h_i denotes the height of $\mathscr{C}_i$), $\mathscr{C}_{i,(a,b)}$ is the set of points in $\mathscr{C}_i$ for which $t_v \in (a, b) \subset (0, h_i)$. Since $\mathscr{C}_{i,(a,b)}$ is $u_t = \left(\begin{smallmatrix} 1 & t \\ 1 & 0 \end{smallmatrix}\right)$ invariant we can define the following asymptotic constants for *finite orbit* $(S, \alpha) \in \mathscr{F}_\tau$:

$$c_{i,(a,b)}(\alpha) = \frac{1}{|\mathscr{O}_\alpha|}\sum_{k=1}^{n_i} \frac{|\mathscr{O}_\alpha \cap \mathscr{C}_{i,(a,b)}|}{w^2_{i,k}}, \tag{35}$$

and for *generic* $(S, \alpha) \in \mathscr{F}_\tau$:

$$c_{i,(a,b)}(\alpha) = \sum_{i=1}^{n_{\mathscr{F}_\tau}} \frac{\operatorname{area}(\mathscr{C}_{i,(a,b)})}{\operatorname{area}(\mathscr{F}_\tau)} \sum_{k=1}^{n_i} \frac{1}{w^2_{i,k}}, \tag{36}$$

The cylinders $\mathscr{C}_{i,k}$ contained in the horizontal foliation of $(S, \alpha) \in \mathscr{C}_{i,(a,b)}$ have heights restricted by the condition $t_v \in (a, b)$. The condition also sets a restriction on the vertical distance of the two cone points of (S, α). This in turn implies that the *areas* of the horizontal cylinders for $(S, \alpha) \in \mathscr{C}_{i,(a,b)}$ are restricted. Since the cylinder area is invariant under $\mathrm{SL}_2(\mathbb{R})$ deformations the asymptotic constants $c_{i,(a,b)}(\alpha)$ in 35 and 36 are the asymptotic quadratic growth rates of cylinders in $(X, \beta) \in \mathscr{F}_\tau$ which map to horizontal cylinders of some $(S, \alpha) \in \mathscr{C}_{i,(a,b)}$ for certain $A \in \mathrm{SL}_2(\mathbb{Z})$.

For precise statements one can restrict to concrete examples. Note however that for finite orbit (S, α) one can always choose intervals (a, b) such that

$$\mathscr{O}_\alpha \cap \mathscr{C}_{i,(a,b)} = \emptyset,$$

in particular there are no cylinders on differentials $(Y, \tau) \in \mathscr{O}_\alpha$ which have area below a certain positive number.

5. d-symmetric elliptic differentials

A cone point s of order o on (X, ω) is called *degenerated*, if there is an open neighborhood U of s such that $U \backslash \{s\}$ is homeomorphic to the disjoint union of at least two punctured discs. We call s *totally degenerated* if there is a neighborhood U such that $U \backslash \{s\}$ is homeomorphic to o punctured discs. A differential (X, ω) is called degenerated, if it contains degenerated singular

points. We call (X,ω) *totally degenerated* if *all* singular points are totally degenerated.

Note $U\backslash\{s\}$ is homotopy equivalent to a union of circles. Moreover degenerated surfaces are not described by a one form alone, one eventually has to provide information about identifications of (some) points. Degenerated surfaces appear naturally after collapsing cone points. In particular degenerated surfaces are part of the *closure of moduli space fibers* $\mathscr{F}$.

d-symmetric torus coverings. We consider elliptic differentials of the shape

$$(X,\omega) = (\#^d_{t_h+it_v}\mathbb{T}^2, \#^d_{t_h+it_v} dz)$$

with a line segment $[0,1]\cdot(t_h+it_v) \subset \mathbb{C}$. Note that $\mathbb{Z}/d\mathbb{Z}$ acts on $\#^d_{t_h+it_v}\mathbb{T}^2$ as subgroup of

$$\mathrm{Aut}(\#^d_{t_h+it_v}\mathbb{T}^2) := \mathrm{Aut}(\#^d_{t_h+it_v}\mathbb{T}^2, \#^d_{t_h+it_v} dz), \tag{37}$$

hence the differentials $\#^d_{t_h+it_v}\mathbb{T}^2$ are d-symmetric. Recall that we have already fixed an ordering of the sheets $\mathbb{T}^2$ of $\pi : \#^d_{t_h+it_v}\mathbb{T}^2 \to \mathbb{T}^2$ consistent with the cut and paste along $v = t_h + it_v$ and the action of $\mathbb{Z}/d\mathbb{Z}$, like in figure 5 below.

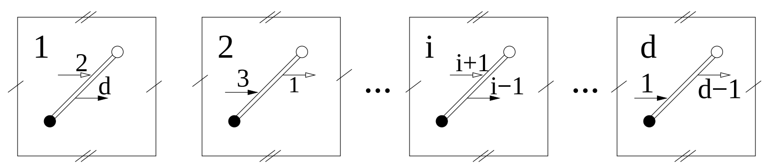

FIGURE 1. Canonical model of a d-symmetric differential

Denote the image of the horizontal line $\{\mathrm{Im}\, z = s\} \subset \mathbb{C}$ on the base torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ by $\mathscr{L}_h(s)$. If the geodesic segment $t_h + it_v \subset \mathbb{T}^2$ is defined as the projection of the geodesic segment connecting 0 and $t_h + it_v$ in $\mathbb{C}$, it intersects $\mathscr{L}_h(s) \in \mathscr{F}_h(\mathbb{T}^2)$ exactly

- $[t_v] + 1$ *times, if* $\{s\} \le \{t_v\}$
- $[t_v]$ *times, if* $\{s\} > \{t_v\}$.

Now parameterize $\mathscr{L}_h(s)$ by the curve

$$\gamma_s : \begin{cases} [0,1] & \to & \mathscr{L}_h(s) \subset \mathbb{T}^2 \\ t & \mapsto & t + is + \mathbb{Z}\oplus\mathbb{Z}i \end{cases}$$

Because of the cut and paste along $t_h + it_v$, the lift $\gamma_{s,i}$ of γ_s to $\#^d_{t_h+it_v}\mathbb{T}^2$, characterized by $\gamma_{s,i}(0) \in \mathbb{T}^2_i$, is on the sheet $\mathbb{T}^2_{i+j}$ when its projection γ_s has crossed the segment $t_h + it_v$ j times. Given a real number a we write by slight abuse of notation $[a]$ for the integer part of a and $\{a\} := a - [a]$. Using this we observe that after one loop of γ_s,

either $\quad \gamma_{s,i}(1) \in \mathbb{T}^2_{i+[t_h]+1}$, if $\{s\} \le \{t_h\}$

or $\qquad \gamma_{s,i}(1) \in \mathbb{T}^2_{i+[t_h]}, \qquad \text{if } \{s\} > \{t_h\}.$

If γ_1 and γ_2 are two chains of geodesic segments on an Abelian differential (X, ω) with

$$\partial\gamma_1 = \partial\gamma_2$$

we might use cut and paste to see that

$$\#^d_{\gamma_1}(X, \omega) = \#^d_{\gamma_2}(X, \omega)$$

if the lifts of γ_1 and γ_2 to the universal covering $(\widetilde{X}, \widetilde{\omega})$ of (X, ω) bound a flat disc (i.e. a disc without cone points). Now the line segment $t_h + it_v \in \mathbb{C}$ decomposes into segments $[t_h] \cdot 1$, $[t_v] \cdot i$ and $\{t_h\} + i\{t_b\}$.

Lemma 1. *Let $\mathscr{F}^{sym}_{d,\mathbb{C}}$ be the component of the fiber of elliptic differentials containing $\{\#^d_{t_h+it_v}(\mathbb{T}^2, dz) : t_h + it_v \in \mathbb{C}\}$. Then any differential contained in $\mathscr{F}^{sym,c}_{d,\mathbb{C}} - \mathscr{F}^{sym}_{d,\mathbb{C}}$ is totally degenerated.*

Proof. Note, that $\mathscr{F}^{sym}_{d,\mathbb{C}}$ is connected, because $\mathbb{C}$ is. Take the d-symmetric differential $\#^d_{h+iv+\{t_h\}+i\{t_v\}}(\mathbb{T}^2, dz)$ associated to the line segment

$$t_h + it_v = [t_h] \cdot 1 + [t_v] \cdot i + \{t_h\} + i\{t_v\} = h \cdot 1 + v \cdot i + \{t_h\} + i\{t_v\}$$

with two integers h and v. Denote the j-th sheet of $\#^d_{h+iv+\{t_h\}+i\{t_v\}}\mathbb{T}^2$ by $\mathbb{T}^2_j$, $j = 1, ..., d$ and rewrite the line segment $h \cdot 1 + v \cdot i + \{t_h\} + i\{t_v\}$ on each $\mathbb{T}^2_i$ as the composition of the three segments $h \cdot 1 \subset \mathbb{T}^2$, $v \cdot i \subset \mathbb{T}^2$ and $\{t_h\} + i\{t_v\} \subset \mathbb{T}^2$. By the discussion before the theorem we jump from the sheet $\mathbb{T}^2_j$ to the sheet $\mathbb{T}^2_{j\pm h(d)}$ when crossing $h \cdot 1 \subset \mathbb{T}^2$ *upwards* $(+)$ or *downwards* $(-)$. If we cross $v \cdot i \subset \mathbb{T}^2$ we jump from $\mathbb{T}^2_j$ to the sheet $\mathbb{T}^2_{j\pm v(d)}$ depending if we cross *to the right* $(+)$ or *to the left* $(-)$.

With fixed $h, v \in \mathbb{Z}$, we now assume $\{t_h\} + i\{t_v\} = 0 \in \mathbb{T}^2$ and follow a circular path $\epsilon e^{i\theta}$ on $\#^d_{h+iv}\mathbb{T}^2$ centered at $0 \in \mathbb{T}^2_j$ counter-clockwise. Starting with small θ we jump to the sheet $j-v$ (d) when crossing the vertical $i{\cdot}v \subset \mathbb{T}^2_j$ to the left. Next we are crossing the horizontal $h \subset \mathbb{T}^2_{j-v\ (d)}$ downwards to jump onto the sheet $j - v - h$ (d). Continuing in that manner we jump back to $j - v$ (d) and then to j (d) when completing the circle at $\theta = 2\pi$. This already shows that all degenerated d symmetric covers $\#^d_{h+iv}(\mathbb{T}^2, dz)$ $(h, v \in \mathbb{Z})$ are in fact totally degenerated.

Now consider a d-symmetric differential defined by the geodesic segments $h \in \mathbb{Z}$, $iv \in i\mathbb{Z}$ and $\epsilon e^{i\theta}$, with $\epsilon > 0$. Changing $\theta \in [0, 2\pi)$ as above gives a closed loop, isometrically embedded in $\mathscr{F}^{sym}_{d,\mathbb{C}}$ and shows that $\mathscr{F}^{sym,c}_{d,\mathbb{C}}$ has no cone points. $\square$

Lemma 2. *Assume the moduli space $\mathscr{E}^{sym}_d$ is connected, then the connected components of $\mathscr{F}^{sym}_d \subset \mathscr{E}^{sym}_d$ have no cone-points of positive order and*

$\mathrm{SL}_2(\mathbb{Z}) \cdot \mathscr{F}^{sym,c}_{d,\mathbb{C}}$ *is isomorphic to a disjoint union of tori:*

$$\mathrm{SL}_2(\mathbb{Z}) \cdot \mathscr{F}^{sym,c}_{d} \cong \mathrm{SL}_2(\mathbb{Z}) \cdot \mathbb{C}/d_1\mathbb{Z} \oplus d_2\mathbb{Z}i \subset \mathscr{E}^{sym,c}_{d}.$$

If $\mathscr{F}^{sym,c}_{d}$ *is connected there is an* $m \in \mathbb{N}$*, such that:*

$$\mathscr{F}^{sym,c}_{d} \cong \mathbb{T}^2_m := \mathbb{C}/m\mathbb{Z} \oplus m\mathbb{Z}i.$$

Proof. The first statement follows immediately from Lemma 1: all connected components of $\mathrm{SL}_2(\mathbb{Z}) \cdot \mathscr{F}^{sym,c}_{d}$ must be tori, because they are in the $\mathrm{SL}_2(\mathbb{Z})$ orbit of $\mathscr{F}^{sym,c}_{d,\mathbb{C}}$ by connectedness of $\mathscr{E}^{sym}_{d}$, hence affine images of the torus $\mathscr{F}^{sym,c}_{d,\mathbb{C}}$. In other words

$$\mathrm{SL}_2(\mathbb{Z}) \cdot \mathscr{F}^{sym,c}_{d} \cong \bigcup_{i=1}^{n} \mathbb{C}/\Lambda(v_i, w_i)$$

with lattices $\Lambda(v_i, w_i)$ generated by two integer vectors $v_i, w_i \in \mathbb{Z}^2$. Because

$$\mathbb{C}/\Lambda(v_i, w_i) \cong A_i \cdot \mathbb{R}^2/a\mathbb{Z} \oplus b\mathbb{Z}$$

for some $A_i \in \mathrm{SL}_2(\mathbb{Z})$ all tori have the same area. The second statement is clear, because rotation by 90 degrees is contained in $\mathrm{SL}_2(\mathbb{Z})$. □

The following Lemma tells us that surfaces in $\mathscr{F}^{sym,c}_{d}$ are actually belonging to $\mathscr{F}^{sym,c}_{d,\mathbb{C}}$.

Lemma 3. *Assume* (X, ω) *is* d*-symmetric and* $\mathrm{Per}(\omega) = \mathbb{Z}^2$.
Then there is a $t_h + it_v \in \mathbb{C}$ *such that*

$$(X, \omega) \cong \#^d_{t_h+it_v}(\mathbb{T}^2, dz).$$

Proof. Since (X, ω) is d-symmetric, we have two covering maps:

$$\pi_d : X \to X/(\mathbb{Z}/d\mathbb{Z}) \cong \mathbb{C}/\Lambda \quad \text{and} \quad \mathrm{pr} : X \to \mathbb{C}/\mathrm{Per}(\omega) \cong \mathbb{T}^2.$$

This implies there is a translation covering $\mathbb{T}^2 \to X/(\mathbb{Z}/d\mathbb{Z})$ and by the cone point orders of (X, ω) we must have $X/(\mathbb{Z}/d\mathbb{Z}) \cong \mathbb{T}^2$. We assume one cone point on (X, ω), say $z_0 \in X$ is the preimage of $[0] \in \mathbb{T}^2$. If the second cone point is the preimage of $[a + ib] \in \mathbb{T}^2$ we remove the line segment $I_{a+ib} := \{t[a + ib] : t \in [0, 1]\} \subset \mathbb{T}^2$. We can present $X - \pi_d^{-1}(I_{a+ib})$ by d squares

$$\mathscr{Q}_{a+ib,j} := [0, 1] \times [0, i] - I_{a+ib} \subset \mathbb{C} \quad j = 1, ..., d$$

with *certain identifications along the boundaries* of $\partial\mathscr{Q}_{a+ib,j}$. Further we can assume $\mathbb{Z}/d\mathbb{Z}$ acts by moving $\mathscr{Q}_{a+ib,j}$ to $\mathscr{Q}_{a+ib,j+1 \mod d}$.

Denote the four boundary line segments of $\mathscr{Q} = [0, 1] \times [0, i]$ by $\partial^*\mathscr{Q}$ with $* = btm$ for the "bottom" component, $* = top$ for the "top", $* = left$ for the "left" and $* = right$ for the "right" boundary component. Since the identifications done along the boundary and along I_{a+ib} have to match with the $\mathbb{Z}/d\mathbb{Z}$-action we must have

- $\partial^{btm}\mathscr{Q}_{a+ib,j} = \partial^{top}\mathscr{Q}_{a+ib,j+k}$

- $\partial^{left}\mathscr{Q}_{a+ib,j} = \partial^{right}\mathscr{Q}_{a+ib,j+l}$

for all $j = 1, ..., d$. By choice when crossing I_{a+ib} from the left, we change from $\mathscr{Q}_{a+ib,j}$ to $\mathscr{Q}_{a+ib,j+1}$. Now given two numbers $k, l \in \{0, ..., d\}$, the line segment $l+ik+a+ib \in \mathbb{C}$ defines a d-symmetric covering $\#^d_{(a+l)+i(b+k)}(\mathbb{T}^2, dz)$ which has all the above properties and thus

$$(X, \omega) \cong \#^d_{(a+l)+i(b+k)}(\mathbb{T}^2, dz).$$

$\square$

Proof of Theorem 4. By Lemma 3 $\mathscr{F}^{sym}_d$ is connected and thus a torus $\mathbb{C}/\Lambda$ by Lemma 1. The lattice Λ of this torus is a sub-lattice of $\mathbb{Z} \oplus \mathbb{Z}i$ by Lemma 2, moreover by $\mathrm{SL}_2(\mathbb{Z})$-invariance of $\mathscr{F}^{sym}_d$ (rotation by 90 degrees!), $\Lambda = m\mathbb{Z} \oplus m\mathbb{Z}i$ for an $m \in \mathbb{N}$. Thus

$$\mathscr{F}^{sym}_d \cong \mathbb{C}/m\mathbb{Z} \oplus m\mathbb{Z}i$$

for an $m \in \mathbb{N}$. Now for $1 > \epsilon > 0$ take the horizontal path

$$\gamma : t \longmapsto \left[\#^d_{t+i\epsilon}\mathbb{T}^2\right] \subset \mathscr{F}_h(\mathscr{F}^{sym}_d),$$

then using Lemma 3 it is easy to see that $\gamma(0) = \gamma(d)$. One can argue as follows: split the line segment $d + i\epsilon$ into two segments d and $i\epsilon$ and consider the surface $\#^d_{d+i\epsilon}\mathbb{T}^2$. The monodromy of $\#^d_{d+i\epsilon}\mathbb{T}^2$ agrees with the monodromy of $\#^d_{i\epsilon}\mathbb{T}^2 = \#^d_{0+i\epsilon}\mathbb{T}^2$ and thus $\#^d_{d+i\epsilon}\mathbb{T}^2 \cong \#^d_{0+i\epsilon}\mathbb{T}^2$, hence $m = d$. $\square$

Twist coordinates on $\mathbb{T}^2_d$. For $1 > t_v > 0$ consider the d-symmetric differential $\#^d_{t_h+it_v}\mathbb{T}^2$. Then the horizontal foliation of $\#^d_{t_h+it_v}\mathbb{T}^2$ contains $d+1$ cylinders: d of width 1 attached to the top of one cylinder of width d. We call t_h the *horizontal twist* and t_v the *vertical twist* of $\#^d_{t_h+it_v}\mathbb{T}^2$. We just found, that parameterization of d-symmetric differentials by horizontal- and vertical twists gives an isometric universal covering

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{T}^2_d \\ t_h + it_v &\longmapsto \left[\#^d_{t_h+it_v}\mathbb{T}^2\right] \end{aligned} \tag{38}$$

Global description of $\mathscr{E}^{sym}_d$. To describe the space $\mathscr{E}^{sym}_d$ of all d-symmetric differentials (with fixed area d), we note that there is a fibration

$$\mathscr{F}^{sym}_d \longrightarrow \mathscr{E}^{sym}_d \longrightarrow \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}). \tag{39}$$

Since $\mathscr{F}^{sym}_d \cong \mathbb{T}^2_d$, we can describe $\mathscr{E}^{sym}_d$ as

$$\mathscr{E}^{sym}_d \cong \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{C}_{\mathbb{Z}\oplus\mathbb{Z}i}/\mathrm{SL}_2(\mathbb{Z}) \ltimes d(\mathbb{Z} \oplus \mathbb{Z}i). \tag{40}$$

with $\mathbb{C}_{\mathbb{Z}\oplus\mathbb{Z}i} := \mathbb{C} - \mathbb{Z} \oplus \mathbb{Z}i$. Note: for all d-symmetric differentials $\#^d_{t_h+it_v}\mathbb{T}^2$ there is an involution $\phi \in \mathrm{Aff}^+(\#^d_{t_h+it_v}\mathbb{T}^2)$, i.e. ϕ is affine linear with $\mathrm{D}\,\phi = -\,\mathrm{id}$. ϕ exchanges the two cone-points of order d on $\#^d_{t_h+it_v}\mathbb{T}^2$. Consequently classifying d-symmetric differentials without *named* cone-points gives a sphere (lattice points removed) $\mathbb{CP}^1_{d,\mathbb{Z}\oplus\mathbb{Z}i} := \mathscr{F}^{sym}_d/(-\,\mathrm{id}) = \mathbb{T}^2_{d,\mathbb{Z}\oplus\mathbb{Z}i}/(-\,\mathrm{id})$,

which admits only a *quadratic differential*. The $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{F}_d^{sym}$ descends to an $\mathrm{PSL}_2(\mathbb{R})$ action on $\mathbb{CP}^1_{d,\mathbb{Z}\oplus\mathbb{Z}i}$ and the moduli space $\mathscr{E}^{sym}_{d,\pm}$ has the following fiber bundle structure:

$$\mathbb{CP}^1_{d,\mathbb{Z}\oplus\mathbb{Z}i} \longrightarrow \mathscr{E}^{sym}_{d,\pm} \longrightarrow \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSL}_2(\mathbb{Z}) \tag{41}$$

Forgetting the elliptic differential structure of $(\#^d_{t_h+it_v}\mathbb{T}^2, \#^d_{t_h+it_v}dz)$ gives the moduli space $\mathscr{M}_d^{sym}$ d-symmetric torus covers of genus d with two branch points of order $d-1$. Algebraically that is dividing out the circle bundle defined by the action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathscr{E}_d^{sym}$

$$\mathbb{CP}^1_{d,\mathbb{Z}\oplus\mathbb{Z}i} \longrightarrow \mathscr{M}_d^{sym} \longrightarrow \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}\ ). \tag{42}$$

6. Geometric properties of d-symmetric differentials

Now we compose our knowledge about the modular fiber $\mathbb{T}^2_d$ of d symmetric torus covers over the base lattice $\mathbb{Z}\oplus\mathbb{Z}i$ and our counting formula to find the quadratic growth rate function on $\mathbb{T}^2_d$.

Proposition 5. *Take the differential $\#^d_{t_h+it_v}\mathbb{T}^2$ and assume $0 < t_h < 1$. Then the horizontal foliation of $\#^d_{t_h+it_v}\mathbb{T}^2$ contains*

- $([t_v], d)$ *cylinders of width* $\frac{d}{([t_v],d)}$ *and*
- $([t_v]+1, d)$ *cylinders of width* $\frac{d}{([t_v]+1,d)}$, *if* $t_v \notin \mathbb{Z}$
- $([t_v], d)$ *cylinders of width* $\frac{d}{([t_v],d)}$, *if* $t_v \in \mathbb{Z}$.

We use $(0, d) = d$ *here.*

Proof. By the discussion before the Proposition the point $\gamma_{i,s}(1)$ is in $\mathbb{T}^2_{i+[t_v]+1}$, if $s \mod 1 \leq t_h \mod 1$ and in $\mathbb{T}^2_{i+[t_v]}$, if $s \mod 1 > t_h \mod 1$. By $\mathbb{Z}/d\mathbb{Z}$ symmetry continuing $\gamma_{i,s}$ we are back on the leaf i as soon as $n([t_v]+1) \equiv 0 \mod d$ in the first case and $n \cdot [t_v] \equiv 0 \mod d$ in the second. That means the cylinder $\mathscr{C}$ containing $\gamma_{i,s}$ has width $w = \frac{d}{([t_v]+1,d)}$ if $s \mod 1 \leq t_v \mod 1$ and width $w = \frac{d}{([t_v],d)}$ if $s \mod 1 < t_v \mod 1$. In case $([t_v]+1, d) > 1$ or $([t_v], d) > 1$ there are sheets $\mathbb{T}^2_k$ which $\mathscr{C}$ does not touch. Using the $\mathbb{Z}/d\mathbb{Z}$ action we find all other horizontal cylinders containing the preimages of the curve γ_s. Since the total length of all the preimages of γ_s is d, there are exactly $([t_v]+1, d)$ cylinders $\mathscr{C}$ above γ_s in the first case and $([t_v], d)$ in the second. □

The horizontal foliation of $\mathscr{F}_d^{sym}$ decomposes into d open cylinders $\mathscr{C}_0, ..., \mathscr{C}_{d-1}$, each of width d and height 1. The surface $\#^d_{t_h+it_v}\mathbb{T}^2$ belongs to $\mathscr{C}_i$, if and only if $[t_v] = i$ and $t_v \notin \mathbb{Z}$. If $t_v = i \in \mathbb{Z}$,

$$\#^d_{t_h+it_v}\mathbb{T}^2 \in \partial^{top}\mathscr{C}_{i-1} = \partial^{btm}\mathscr{C}_i.$$

Theorem 6. *The horizontal foliation of $(X, \omega) \in \mathscr{F}_d^{sym}$ contains*

- *(i,d) cylinders of width $\frac{d}{(i,d)}$ and*
- *$(i+1,d)$ cylinders of width $\frac{d}{(i+1,d)}$, if $(X,\omega) \in \mathscr{C}_i$*
- *(i,d) cylinders of width $\frac{d}{(i,d)}$, if $(X,\omega) \in \partial^{btm}\mathscr{C}_i$.*

Proof. The number and width of closed cylinders in the horizontal foliation of $X = \#^d_{t_h+it_v}\mathbb{T}^2$ are the same for all $X \in \mathscr{C}_i \subset \mathscr{F}_h(\mathscr{F}_d^{sym})$. The statement now follows from Proposition 5. □

The vertical foliation $\mathscr{F}_v(\mathscr{F}_d^{sym})$ admits a decomposition into open cylinders $\mathscr{C}_{v,0},...,\mathscr{C}_{v,d-1}$ (of width d and height 1) and boundary components $\partial^{btm}\mathscr{C}_{v,0},... ...,\partial^{btm}\mathscr{C}_{v,d-1}$ as well and we have:

Corollary 2. *The vertical foliation of $(X,\omega) \in \mathscr{F}_d^{sym}$ contains*

- *(i,d) cylinders of width $\frac{d}{(i,d)}$ and*
- *$(i+1,d)$ cylinders of width $\frac{d}{(i+1,d)}$, if $(X,\omega) \in \mathscr{C}_i$*
- *(i,d) cylinders of width $\frac{d}{(i,d)}$, if $(X,\omega) \in \partial^{btm}\mathscr{C}_i$.*

Proof. To prove this statement, we rotate the whole space $\mathbb{T}^2_d = \mathscr{F}_d^{sym}$ by

$$r_{\pi/2} := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}).$$

That maps the surface $\#^d_{t_h+it_v}\mathbb{T}^2 \in \mathbb{T}^2_d$ to

$$r_{\pi/2} \cdot \#^d_{t_h+it_v}\mathbb{T}^2 = \#^d_{t_v-it_h}\mathbb{T}^2 \in \mathbb{T}^2_d$$

and isometrically maps the horizontal foliation of $\#^d_{t_h+it_v}\mathbb{T}^2$ to the vertical foliation of $\#^d_{t_v-it_h}\mathbb{T}^2$. On the level of $\mathscr{F}_d^{sym}$ $r_{\pi/2}$ maps the horizontal cylinder $\mathscr{C}_i$ to $\mathscr{C}_{v,d-i}$. Now notice that differentials in $\mathscr{C}_i$ and $\mathscr{C}_{d-i}$ just differ by the (hyperelliptic) involution $-\mathrm{id}$. That proves the claim. □

Note that for prime $d > 2$ there are only 2 possible numbers of *horizontal cylinders* on differentials in the $\mathscr{C}_i$, namely 2 and $d+1$.

Orbits of torsion points. If the *prime-factor decomposition* of $n \in \mathbb{Z}$ is $n = p_1^{l_1} \cdot p_2^{l_2} \dots p_r^{l_r}$, then the number of positive *divisors* $D(n)$ of n is

$$D(n) = (l_1+1)\dots(l_r+1)$$

and we have

Proposition 6. *The set $\mathbb{T}^2_d[m]$ on $\mathbb{T}^2_d \cong \mathscr{F}_d^{sym,c}$ contains precisely $D(m)$ $\mathrm{SL}_2(\mathbb{Z})$-orbits. In particular the set of degenerated differentials, represented by $\mathbb{T}^2_d[d] \subset \mathscr{F}_d^{sym,c}$, is the disjoint union of $D(d)$ $\mathrm{SL}_2(\mathbb{Z})$-orbits.*

We call the periodic foliation $\mathscr{F}_\theta(\#^d_v\mathbb{T}^2)$ of a d-symmetric differential $\#^d_v\mathbb{T}^2$ *simple* if every cylinder $\mathscr{C}_j$ in $\mathscr{F}_\theta(\#^d_v\mathbb{T}^2)$ is in the $\mathbb{Z}/d\mathbb{Z}$ orbit of any other cylinder $\mathscr{C}_i$.

Theorem 7. *For $v \in \mathbb{T}^2_d \cong \mathscr{F}_d^{sym}$ the following statements are equivalent:*

1. *The d-symmetric differential $\#_v^d\mathbb{T}^2$ contains a simple direction with precisely $k|d$ closed cylinders.*
2. *There is an $A \in \mathrm{SL}_2(\mathbb{Z})$, and an $m \in \mathbb{Z}/d\mathbb{Z}$, such that*

$$[A \cdot v] \in \partial^{top}\mathscr{C}_m \text{ with } k = (m, d).$$

Moreover, if $v \in \mathbb{T}_d^2$ is a torsion point, we can add the statement:

3. *There is an $n \in \mathbb{N}$ with $\frac{d}{(d,m)}|n$, and $v \in \mathbb{T}_d^2(n)$.*

Proof. By Theorem 6 the horizontal foliation of a d-symmetric surface $\#_v^d\mathbb{T}^2$ contains precisely $k|d$ cylinders if $v \in \partial^{top}\mathscr{C}_m$ with $k = (m, d)$. Now if a simple, periodic foliation of $\#_v^d\mathbb{T}^2$ contains precisely $k|d$ cylinders there is an $A \in \mathrm{SL}_2(\mathbb{Z})$ making this direction horizontal and thus we must have $A \cdot v \in \partial^{top}\mathscr{C}_m$ with $k = (m, d)$.

Finally we need to find $A \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{bmatrix} * \\ m/d \end{bmatrix} = A \cdot \begin{bmatrix} 1/n \\ 0 \end{bmatrix} \quad \text{on } \mathbb{T}^2.$$

Note that $k = (m, d)|d$. Since $\mathscr{O}_n = \mathrm{SL}_2(\mathbb{Z}) \cdot [1/n, 0] = \mathbb{T}^2(n)$, we have the condition $d/k = d/(m, d)|n$. $\square$

Remark: Suppose $\mathscr{F}$ is a 2-dimensional modular fiber of $\mathbb{T}^2$ coverings and the natural projection $\pi_* : \mathscr{F} \to \mathbb{T}^2$ factors over $\mathbb{T}_d^2$, then Proposition 6 implies that the preimage

$$\mathscr{F}[d] := \pi_*^{-1}(\mathbb{T}_d^2[d]) \subset \mathscr{F}$$

of the d-torsion points on $\mathbb{T}_d^2$ splits into at least $D(d)$ different $\mathrm{SL}_2(\mathbb{Z})$ orbits. If $\mathscr{F}$ is a space of differentials with two cone points, the origamis, i.e. surfaces which can be tiled by unit squares, in $\mathscr{F}$ are a subset of $\mathscr{F}[d]$.

We provide examples for this factorization of π_* in [S2, S3]. However for $\mathscr{F}_d(2) \subset \mathscr{F}_d(1,1)$ the splitting of orbits is already known, see [McM4, HL].

Differentials on integer coordinates. We like to describe the degenerated surfaces having integer twist coordinates, $(j, k) \in \mathbb{Z}^2/d\mathbb{Z}^2 \subset \mathscr{F}_d^{sym,c}$. For the first we do not care about, whether some of these degenerated surfaces are isomorphic, we just study their shape and their $\mathrm{SL}_2(\mathbb{Z})$-orbits in $\mathscr{F}_d^{sym,c}$.

Theorem 8. *For $(j, k) \in \mathbb{Z}^2/d\mathbb{Z}^2$ the degenerated surface $\#_{j+ik}^d\mathbb{T}^2$ is a union of $(j, k, d) := \gcd(j, k, d)$ tori, with all integer lattice points identified. Each torus-component contained in $\#_{j+ik}^d\mathbb{T}^2$ has area $d/(j, k, d)$ and contains $(k, d)/(j, k, d)$ horizontal cylinders of width $d/(k, d)$ and $(j, d)/(j, k, d)$ vertical cylinders of width $d/(j, d)$.*

Proof. Look at the surfaces with twist coordinate $j + i0 \in \mathbb{Z}^2/d\mathbb{Z}^2$. The surface $\#_j^d\mathbb{T}^2$ parameterized by $j \in \mathbb{Z}/d\mathbb{Z}$ has d cylinders of width 1 in the horizontal direction and (j, d) vertical cylinders of width $d/(j, d)$. Since $\#_j^d\mathbb{T}^2$ is a one point union of tori, it consists of (j, d) tori each of width 1

and height $d/(j,d)$. Since every differential contained in $\mathbb{Z}^2/d\mathbb{Z}^2$ is in the $\mathrm{SL}_2(\mathbb{Z})$ orbit of one of the surfaces $\#_j^d\mathbb{T}^2$, $j \in \mathbb{Z}/d\mathbb{Z}$ and we know all the degenerated surfaces $\#_{j+ik}^d\mathbb{T}^2$ are on the $\mathrm{SL}_2(\mathbb{Z})$ orbit of $\#_a^d\mathbb{T}^2$ if and only if $(j,k,d) = (a,d)$, all surfaces $\#_{j+ik}^d\mathbb{T}^2$ with fixed (j,k,d) are one point unions of (j,k,d) tori, each of area $d/(j,k,d)$.

The second statement now follows from the fact that $\#_{j+ik}^d\mathbb{T}^2$ admits (k,d) horizontal cylinders of width $d/(k,d)$ and (j,d) vertical cylinders of width $d/(j,d)$ shown in Theorem 6 and Corollary 2. $\square$

Isomorphic surfaces on the integer lattice. Assume $\#_{j+ik}^d\mathbb{T}^2 \in \mathbb{Z}^2/d\mathbb{Z}^2 \in \mathscr{F}_d^{sym,c}$, then

$$\#_{j+ik}^d\mathbb{T}^2 \cong \#_{-j-ik}^d\mathbb{T}^2 = r_\pi \cdot \#_{j+ik}^d\mathbb{T}^2$$

are isomorphic. Mainly because surfaces represented by the integer lattice $\mathbb{Z}^2/d\mathbb{Z}^2$ admit only one cone point and thus the property of distinguishing cone points, which allow to distinguish $\#_{j+ik}^d\mathbb{T}^2$ and $r_\pi \cdot \#_{j+ik}^d\mathbb{T}^2$ disappears.

Splitting of orbits – moduli space recognition of invariant. Recently Hubert and Lelievre [HL] followed by a generalized approach of McMullen [McM4] have described invariants to distinguish $\mathrm{SL}_2(\mathbb{Z})$-orbits of genus 2 elliptic differentials with a single zero (of order 2). We suspect that one can observe a similar phenomen in a more general setup. In particular for differentials in genus 2 with 2 zeros of order 1 [S3, S4].

7. Asymptotic constants — cylinders, generic case

Siegel Veech constants for generic differentials. We write down the Siegel-Veech constants as function of the point in the fiber of the moduli space. As before we restrict our considerations to the fiber $\mathscr{F}_d^{sym}$.

Proof of Theorem 5. Because (X,ω) in not arithmetic, $(X,\omega) \in \mathscr{F}_d^{sym} = \mathbb{T}_d^2$ is not a torsion point and we apply Theorem 5, Formula 3 to obtain the result.

Now the horizontal foliation of $\mathscr{F}_d^{sym}$ decomposes into d open cylinders $\mathscr{C}_1, ..., \mathscr{C}_d$, each of width d and height 1. We can label the cylinders by the (integer) vertical twist $i = [t_v]$. By Theorem 6 we have $(i+1,d)$ cylinders of width $d/(i+1,d)$ and (i,d) cylinders of length $d/(i,d)$. Cases when the vertical twist t_v is integer play no role for the generic constant. Now $i = [t_v]$ is a natural number ranging from 1 to d. Since each strip with integer vertical twist i has area d and $\mathrm{area}(\mathscr{F}_d^{sym}) = d^2$, we find the generic Siegel-Veech constant for cylinders of closed geodesics:

$$c_{gen}(S) \;=\; \sum_{i=1}^{d} \frac{\mathrm{area}(\mathscr{C}_i)}{\mathrm{area}(\mathscr{F}_d^{sym})}\left[\frac{(i,d)^3}{d^2} + \frac{(i+1,d)^3}{d^2}\right] \;=\; \frac{2}{d^3}\sum_{i=1}^{d}(i,d)^3. \tag{43}$$

Now for given $p|d$ the number of $\bar{i} \in \mathbb{Z}/d\mathbb{Z}$ with $p = (i,d)$ is $\varphi(d/p)$ and thus:

$$c_{gen}(S) = \frac{2}{d^3}\sum_{i=1}^{d}(i,d)^3 = 2\sum_{p|d}\frac{\varphi(p)}{p^3}. \tag{44}$$

□

Limit genus to ∞. For genus to infinity we make the following simple limit consideration:

$$\frac{c_{gen}}{d} = \frac{2}{d}\sum_{p|d}\frac{\varphi(p)}{p^3} \leq \frac{2\zeta(2)}{d} \overset{d\to\infty}{\longrightarrow} 0.$$

Of course d-symmetric differentials contain no unfolded rational billiard tables and it is hard to start speculations how their growth rates behave for genus to infinity based on d-symmetric differentials. A second remark: the infinite genus surface $\#_v^\infty \mathbb{T}^2$ has *infinitely many* cylinders of width 1 in the vertical and horizontal direction. Thus one cannot even start a reasonable direct counting of closed cylinders.

8. Asymptotic constants — cylinders, finite orbit case

What is left is to calculate the Siegel-Veech constants for d-symmetric differentials represented by torsion points of order n in $\mathbb{T}^2_d$. For fixed $n \in \mathbb{N}$ take a natural number $1 \leq a \leq n$ and define

$$\frac{\pi}{\zeta(2)}c_{d,n}(a) := \lim_{T\to\infty}\frac{N(V_n(a),T)}{T^2}$$

to be the quadratic growth rate of the distribution

$$\begin{aligned} V_n(a) := \{A\cdot \mathrm{hol}(c) : c \in Cyl(\mathscr{F}_h(\#_v^d\mathbb{T}^2)) \text{ with} \\ v \in \mathscr{L}_{ad/n}\cap \mathbb{T}^2_d(n),\ A \in \mathrm{SL}_2(\mathbb{Z})\}. \end{aligned} \tag{45}$$

Lemma 4. *For $1 \leq a \leq n$ and $t_v(a) := [ad/n] \in \mathbb{Z}$ we have*

$$c_{d,n}(a) = \frac{n}{\varphi(n)\psi(n)}\frac{\varphi((a,n))}{(a,n)}\left(\frac{(t_v(a),d)^3}{d^2} + \frac{(t_v(a)+1,d)^3}{d^2}\right). \tag{46}$$

Proof. The number of torsion points $\mathbb{T}^2_d(n)$ of order n on the horizontal leaf $\mathscr{L}_{ad/n} \subset \mathbb{T}^2_d$ going through the point $[iad/n] \in \mathbb{T}^2_d$ equals

$$\begin{aligned} |\{b \in \mathbb{Z}/n\mathbb{Z} : (b,a,n) = 1\}| &= |\{b \in \mathbb{Z}/n\mathbb{Z} : (b,(a,n)) = 1\}| = \\ &= \begin{cases} |\psi^{-1}((\mathbb{Z}/(a,n)\mathbb{Z})^*)|, & \text{if } (a,n) \geq 2 \\ n, & \text{if } (a,n) = 1 \end{cases} \end{aligned}$$

where $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(a,n)\mathbb{Z}$ is the natural homomorphism. Thus

$$|\mathscr{L}_{ad/n}\cap \mathbb{T}^2_d(n)| = n\frac{\varphi((a,n))}{(a,n)}. \tag{47}$$

By Theorem (6) the horizontal foliation of a differential $\#_w^d\mathbb{T}^2 \in \mathscr{L}_{ad/n} \cap \mathbb{T}_d^2(n)$

- *always contains* $(t_v(a), d)$ *cylinders of width* $d/(t_v(a), d)$
- *and contains* $(t_v(a)+1, d)$ *cylinders of width* $d/(t_v(a)+1, d)$ *if* $t_v(a) \notin \mathbb{Z}$.

With $|\mathbb{T}_d^2(n)| = \varphi(n)\psi(n)$ we find the quadratic growth rates above. □

Remark. The first part of the argument in the proof of the Lemma tells us that the Teichmüller disk through the torsion points $\mathbb{T}^2(n)$ of order n has

$$cu(n) = \frac{1}{2}\sum_{a=1}^{n}\varphi((a,n)) = \frac{1}{2}\sum_{l|n}\varphi\left(\frac{n}{l}\right)\varphi(l) \tag{48}$$

cusps if $n \geq 3$, and 2 cusps if $n = 2$.

Now the Siegel-Veech constant $c_{d,n} = \sum_{a=1}^n c_{d,n}(a)$ for periodic cylinders on d-symmetric differentials $\#_w^d\mathbb{T}^2$ with $[w] \in \mathbb{T}_d^2(n)$ is

$$\begin{aligned} c_{d,n} = \frac{n}{\varphi(n)\psi(n)} \Bigg(& \sum_{a=1}^{n} \frac{\varphi((a,n))}{(a,n)} \frac{(t_v(a),d)^3}{d^2} + \\ & + \sum_{ad/n \notin \mathbb{Z}} \frac{\varphi((a,n))}{(a,n)} \frac{(t_v(a)+1,d)^3}{d^2} \Bigg). \end{aligned} \tag{49}$$

To simplify this expression we **restrict to torsion points of order n with** $(\mathbf{n},\mathbf{d}) = \mathbf{1}$. Then all numbers a such that $\overline{ad/n} \in \mathbb{Z}/d\mathbb{Z}$ are multiples of n, consequently

$$|\mathbb{T}_d^2(n) \cap \partial^{btm}\mathscr{C}_i| = \begin{cases} \varphi(n) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{50}$$

and:

$$c_d(n) = \frac{n}{\varphi(n)\psi(n)} \left(d\frac{\varphi(n)}{n} + 2 \sum_{a \not\equiv 0 \mod n} \frac{\varphi((a,n))}{(a,n)} \frac{(t_v(a),d)^3}{d^2} \right). \tag{51}$$

We always assume $a \in \{-n+1, -n+2, \ldots -1, 0, 1, \ldots, n-1\}$, representing a class $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$. If furthermore **d is prime** the numbers of isotopy classes in a given direction is either $1, 2, d$ or $d+1$ and the quadratic growth rates are:

# of cylinders	$c_d(n) =$
1	$\frac{n}{\varphi(n)\psi(n)} \frac{1}{d^2} \sum_{da/n \in \{1,\ldots,d-1\}} \frac{\varphi((a,n))}{(a,n)}$
2	$\frac{n}{\varphi(n)\psi(n)} \frac{2}{d^2} \sum_{d-1 > \frac{a}{n} d > 1} \frac{\varphi((a,n))}{(a,n)}$
d	$\frac{n}{\varphi(n)\psi(n)} \frac{d\varphi(n)}{n} = \frac{d}{\psi(n)}$
d+1	$\frac{n}{\varphi(n)\psi(n)} 2\left(d + \frac{1}{d^2}\right) \sum_{0<\lvert a\rvert<n/d} \frac{\varphi((a,n))}{(a,n)}$

In particular there are two possibilities for prime d:

— **either** $(n, d) = 1$, then there are *no* directions with only *one isotopy class* on surfaces located on the $\mathrm{SL}_2(\mathbb{Z})$ orbit of $[1/n]$ and we find

$$c_d(n) = \frac{n}{\varphi(n)\psi(n)} \left(\frac{2}{d^2} \sum_{1 < \frac{a}{n} d < d-1} \frac{\varphi((a,n))}{(a,n)} + \frac{d\varphi(n)}{n} + \right. \tag{52}$$
$$\left. + 2\left(d + \frac{1}{d^2}\right) \sum_{0 < \frac{|a|}{n} d < 1} \frac{\varphi((a,n))}{(a,n)} \right),$$

— **or** d divides n, then the asymptotic constant is the sum of all four terms in the table above, in particular there are *always* directions with only *one isotopy class* on surfaces in the orbit of $[1/n]$.

Results for $d = 2$. For small d the above formulas simplify. In particular for $d = 2$ and n *odd*, formula (52) reads

$$c_2(n) = \frac{1}{\varphi(n)\psi(n)} \left(2\varphi(n) + \frac{9n}{4} \sum_{a \not\equiv 0 \mod n} \frac{\varphi((a,n))}{(a,n)} \right) = \tag{53}$$
$$= \frac{2}{\psi(n)} + \frac{9}{4}\left(1 - \frac{1}{\psi(n)}\right) = \frac{9}{4} - \frac{1}{4\psi(n)}.$$

The only thing we have used here is the obvious identity derived from equation (47):

$$\varphi(n)\psi(n) = n \sum_{a=1}^{n} \frac{\varphi((a,n))}{(a,n)} = \varphi(n) + n \sum_{a=1}^{n-1} \frac{\varphi((a,n))}{(a,n)}.$$

Note, if $d = 2$ the d-cylinder directions are in fact 2-cylinder directions. Because $1 \leq a \leq n$ for a system of representants a modulo n there is no $|a| > n/2$. In more geometrical terms: the modular fiber has two horizontal cylinders, and the horizontal foliation of surfaces in both of these cylinders

have $3 = 2 + 1$ isotopy-classes of closed geodesics.

For *even* n the twist $a = n/d = n/2$ is integer and is on the horizontal leaf with vertical twist one. This horizontal leaf (see table above) reflects all directions with exactly one closed cylinder, thus for *even* n we obtain:

$$\begin{aligned} c_2(n) &= \frac{1}{\varphi(n)\psi(n)} \left(2\varphi(n) + \frac{1}{4}\varphi(\frac{n}{2}) + \frac{9n}{2} \sum_{2a/n \neq 1,2} \frac{\varphi((a,n))}{(a,n)} \right) = \\ &= \begin{cases} \frac{9}{4\psi(n)} + \frac{9}{4}\left(1 - \frac{2}{\psi(n)}\right) = \frac{9}{4} - \frac{9}{4\psi(n)} & \text{if } 4 \nmid n \\ \frac{17}{8\psi(n)} + \frac{9}{4}\left(1 - \frac{3}{2\psi(n)}\right) = \frac{9}{4} - \frac{5}{4\psi(n)} & \text{if } 4 \mid n. \end{cases} \end{aligned} \tag{54}$$

We have used

$$\varphi\left(\frac{n}{2}\right) = \begin{cases} \varphi(n), & \text{if } 4 \nmid n \\ \frac{1}{2}\varphi(n), & \text{if } 4 \mid n. \end{cases} \tag{55}$$

9. Asymptotic constants — saddle connections

Since d-symmetric differentials have automorphism group $\mathbb{Z}/d\mathbb{Z}$ and are coverings of degree d, they are are *balanced*. Take $\#^d_{t_h+it_v}\mathbb{T}^2$ d-symmetric and let $\pi : \#^d_{t_h+it_v}\mathbb{T}^2 \to (\mathbb{T}^2, [0], [t_h + it_v])$ be the canonical projection, then any saddle connection $s \in SC(\mathbb{T}^2, [0], [t_h + it_v])$ has d preimages

$$\pi^{-1}(s) = \{s_1, ..., s_d\} = \{\mathrm{Aut}(\#^d_{t_h+it_v}\mathbb{T}^2) \cdot s_1\} \subset SC(\#^d_{t_h+it_v}\mathbb{T}^2).$$

Thus to find the quadratic growth rate of all saddle connections on $\#^d_{t_h+it_v}\mathbb{T}^2$ is easy, if we know the one for the two marked torus $(\mathbb{T}^2, [0], [t_h + it_v])$. We did this in [S1], however it is much easier to use the method developed in this paper. The main thing to do is to find the asymptotic quadratic growth constants for saddle connections on $\mathbb{T}^2$ connecting $[0]$ and $[t_h + it_v]$. Assuming $[t_h + it_v] \in \mathbb{T}^2(n) = \mathrm{SL}_2(\mathbb{Z}) \cdot [1/n]$ formula 5 says mainly we need to measure distance of the points in $\mathrm{SL}_2(\mathbb{Z}) \cdot [1/n] \cap \mathscr{L}_0$ to the origin to the left and right and sum up the squares of their reciprocals. This gives

$$c_\pm(n) = \frac{2n^2}{\varphi(n)\psi(n)} \sum_{(i,n)=1} \frac{1}{i^2} \tag{56}$$

with limit $n \to \infty$ the Siegel-Veech constant for saddle connections connecting the two markings on the generic 2-marked torus:

$$c_\pm = \frac{\pi^2}{3}. \tag{57}$$

We obtain the obvious

Corollary 3. *The asymptotic quadratic growth rate $c^d_\pm(n)$ of saddle connections $SC_\pm(\#^d_{t_h+it_v}\mathbb{T}^2)$ on $\#^d_{t_h+it_v}\mathbb{T}^2 \in \mathscr{O}_n \in \mathscr{F}^{sym}_d$ connecting the two cone points of $\#^d_{t_h+it_v}\mathbb{T}^2$ is*

$$c^d_\pm(n) = d\frac{2n^2}{\varphi(n)\psi(n)}\sum_{(i,n)=1}\frac{1}{i^2}. \tag{58}$$

For infinite orbit surfaces (generic) we obtain:

$$c^d_\pm = d\frac{\pi^2}{3}. \tag{59}$$

This is clearly not surprising, but one can ask for certain subsets of this set of saddle connections, which are connected to the number of connected components a degenerate surface in $\mathbb{Z}^2/d\mathbb{Z}^2 \subset \mathscr{F}^{sym,c}_d$ contains. By $\mathrm{SL}_2(\mathbb{Z})$ invariance of the asymptotic constants we need to consider the $\mathrm{SL}_2(\mathbb{Z})$ orbits on $\mathbb{Z}^2/d\mathbb{Z}^2$. As we saw earlier for $m|d$ each orbit $\mathscr{O}_{d/m}$ is generated by a particular torus $\#_m\mathbb{T}^2 \in \mathbb{Z}^2/d\mathbb{Z}^2$ and hence these orbits are in one to one correspondence to the divisors $m|d$. It follows from Theorem 8 that all degenerated surfaces associated to $m|d$ have m connected components after removing their "cone" point. Moreover all these connected components are tiled by d/m squares and contain an equal amount of horizontal saddle connections. Now we use this observation to define subsets of all saddle connections connecting the 2 cone points of any $(S,\alpha) \in \mathscr{F}^{sym}_d$.

To begin with take the (horizontal) spine $\mathscr{SP}_h \subset \mathscr{F}_h(\mathscr{F}^{sym}_d, \omega_d)$ and look at the saddle connections $\mathscr{SP}_{h,m} \subset \mathscr{SP}_h$ emanating or terminating in one of the points in $\mathscr{O}_m$. A surface (S,α) parameterized by $\mathscr{SP}_{h,m} \subset \mathscr{F}^{sym}_d$ degenerates into $\mathscr{O}_m$ along $\mathscr{SP}_{h,m}$. During such a deformation d horizontal saddle connections of (S,α) connecting the two cone points will degenerate. Since by Theorem (8) the degenerated surface has m connected components, this component can be "cut out" of (S,α), by taking d/m of the d saddle connections away. As a consequence we can organize these d saddle connections into m chains of d/m saddle connections which altogether define a boundary in homology. Given d and $m|d$, we call such a configuration of saddle connections an *m-homologous family.*

Note that this notion is well-defined for non-arithmetic (= generic) surfaces in $\mathscr{F}^{sym}_d$. In particular the Siegel-Veech constant for m-homologous families (one chain) on the generic surface in $\mathscr{F}^{sym}_d$ is

$$c^d_\pm(m) = \frac{\pi^2}{3}\frac{\varphi(d/m)\psi(d/m)}{d\cdot m}. \tag{60}$$

Constants for m-homologous cycles on finite orbit surfaces. Given a $d,n \in \mathbb{N}$, we consider the set of saddle connections on

$$\#^d_{(a+ib)d/n}\mathbb{T}^2 \in \mathscr{O}_n \subset \mathscr{F}^{sym}_d.$$

According to the formulas developed earlier each saddle connection on $\#^d_{(a+ib)d/n}\mathbb{T}^2$ is mapped to a horizontal saddle connection on some $\#^d_{(j+ik)d/n}\mathbb{T}^2 \in \mathscr{O}_n$ for some $A \in \mathrm{SL}_2(\mathbb{Z})$. This A is determined up to a parabolic subgroup generated by $u_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with orbit

$$\{u_1^m \cdot A \cdot \#^d_{(j+ik)d/n}\mathbb{T}^2 : m \in \mathbb{Z}\} \cong \mathbb{Z}/w\mathbb{Z}.$$

The order w of this parabolic group equals the *width* of the cusp it defines in

$$\mathrm{SL}(\#^d_{(j+ik)d/n}\mathbb{T}^2, dz) \subset \mathrm{SL}_2(\mathbb{Z}).$$

Since this width appears as a factor in the formula for the Siegel-Veech constants we can actually talk about the Siegel-Veech constants for saddle connections associated to any particular surface (compare equations 19 and 20)

$$\#^d_{(j+ik)d/n}\mathbb{T}^2 \in \mathscr{O}_n.$$

Of course, one cannot really associate a particular set of saddle connections, i.e. a set of saddle connections with distinguished properties depending on $\mathscr{F}_h(\#^d_{(j+ik)d/n}\mathbb{T}^2, dz)$, to $\#^d_{(j+ik)d/n}\mathbb{T}^2$, since these properties are preserved under the action of $\{u^n : \ n \in \mathbb{Z}\}$. What we do instead is looking at the Siegel-Veech constant associated to the orbit

$$\mathscr{U}_{j,k} = \{u_1^n \cdot \#^d_{j+ik}\mathbb{T}^2 : n \in \mathbb{N}\} \subset \partial^{btm}\mathscr{C}_k.$$

and divide the result by the cusp-width $|\mathscr{U}_{j,k}|$. In that sense we understand the next theorem. Further we note that for a saddle connection $s_{a+ib} \in \mathscr{S}\mathscr{P}_{h,m}$ bounded by $\#_{a+ib}\mathbb{T}^2 \in \mathscr{O}_m$ to the right we have $(a, b, d) = m$, while $(a+1, b, d) = m$, if $\#_{(a+1)+ib}\mathbb{T}^2 \in \mathscr{O}_m$ bounds s_{a+ib} to the left.

Theorem 9. *Assume there is an*

$$\#^d_{t_h+it_v}\mathbb{T}^2 \in s_{a+ib} \cap \mathscr{O}_n, \quad \text{with } s_{a+ib} \in \mathscr{S}\mathscr{P}_{h,m}.$$

Then the asymptotic quadratic growth rate of the m-homologous families of saddle connections, which are associated to $\#^d_{t_h+it_v}\mathbb{T}^2$ in the sense above, contained in $(S, \alpha) \in \mathscr{O}_n$ is:

$$c^d_{a,b,m}(\alpha) = \frac{d}{m \cdot |\mathscr{O}_n| \cdot |\{t_h\} - \epsilon|^2}. \tag{61}$$

Here $\epsilon = 1$, if we count saddle connections in the orbit of the one defined by the horizontal segment connecting $\#^d_{t_h+it_v}\mathbb{T}^2$ with $\#^d_{(a+1)+ib}\mathbb{T}^2 \in \mathscr{O}_m$ and $\epsilon = 0$ otherwise. The quadratic growth rate of all m-homologous saddle connections on any $(S, \alpha) \in \mathscr{O}_n$ is:

$$c^d_m(\alpha) = \frac{2 \cdot d}{m \cdot |\mathscr{O}_n|} \sum_{\#^d_{t_h+it_v}\mathbb{T}^2 \in \mathscr{S}\mathscr{P}_{h,m} \cap \mathscr{O}_n} \frac{1}{|\{t_h\}|^2}. \tag{62}$$

Note: Any degeneration of $\#^d_{t_h+it_v}\mathbb{T}^2 \in \mathscr{F}^{sym}_d$ collapsing the two cone points also collapses exactely d saddle connections.

Proof. Straight forward application of formula 5. □

Like for cylinders the constants for saddle connections become easier if we restrict to torsion points of order $n \neq d$, with $(n,d) = 1$, because in this case (see equation 50)

$$\mathbb{T}^2_d(n) \cap \mathscr{S}\mathscr{P}_{h,m} = \mathbb{T}^2_d(n) \cap \partial^{btm}\mathscr{C}_0.$$

Thus we find for $\#^d_{dj/n}\mathbb{T}^2 \in \mathbb{T}^2_d(n) \cap \partial^{btm}\mathscr{C}_0$

$$c^d_{[dj/n]}(dj/n) = \frac{1}{d \cdot m}\frac{n^2}{\phi(n)\psi(n)}\frac{1}{|\{j\}|^2}.$$

Taking the limit $n \to \infty$ while assuming $(n,d) = 1$, gives the generic constants:

$$c^d_{j,0,\pm} = \frac{\pi^2}{3}\frac{1}{d \cdot m}.$$

Continuity of Siegel-Veech constants for $d = 2$. We discuss convergence of Siegel-Veech constants of saddle connections connecting the two cone points in case of 2-symmetric differentials. In this case the modular fiber is presented by

$$\mathscr{F}^{sym}_2 \cong \mathbb{C}/2\mathbb{Z}^2 - \mathbb{Z}^2/2\mathbb{Z}^2.$$

The group $\mathbb{Z}^2/2\mathbb{Z}^2$ parameterizes four degenerated surfaces and using the *multiplication* isomorphism

$$m_2 : \mathbb{T}^2 \to \mathbb{C}/2\mathbb{Z}^2 \cong \mathscr{F}^{sym,c}_2,$$

the three non-vanishing 2-torsion points on $\mathbb{T}^2$ represent one $\mathrm{SL}_2(\mathbb{Z})$ orbit, the other is the point $[0] \in \mathbb{T}^2 \cap \frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$. $\mathscr{F}_h(\mathscr{F}^{sym}_2, \omega_2)$ contains 4 saddle connections $s^\pm_0, s^\pm_1$, where the s^+_i connect $[i/2, 0] \in \mathbb{T}^2$ with $[i/2, 1/2] \in \mathbb{T}^2$ and the s^-_i connect $[i/2, 1/2] \in \mathbb{T}^2$ with $[i/2, 0] \in \mathbb{T}^2$. Now for all $n \in \mathbb{N}$ with $(n, 2) = 1$, we have

$$\mathscr{O}_n \cap s^\pm_1 = \{[a/n, b/n] \in \mathbb{T}^2 : (a, b, n) = 1\} \cap s^\pm_1 = \emptyset.$$

The Siegel-Veech constant $c^\pm_0(n)$ associated with $[0] \in \mathbb{T}^2$ is not affected by this, since $[0]$ is a fixed point of $\mathrm{SL}_2(\mathbb{Z})$. By the above the Siegel-Veech constant $c^\pm_1(n)$ for surfaces on the orbit $\mathscr{O}_n$, n odd, is entirely determined by intersections $\mathscr{O}_n \cap s^\pm_0$. In fact, one easily states

$$c^\pm_1(n) = 4\frac{n^2}{\varphi(n)\psi(n)}\sum_{\substack{i=1\\(i,2n)=1}}^{(n+1)/2}\frac{1}{i^2}. \tag{63}$$

Now taking the limit over any sequence of *odd* numbers we find

$$c^\pm_1 = 4\frac{\varphi(2)\psi(2)}{4}\zeta(2) = 3 \cdot \zeta(2). \tag{64}$$

To see this, one can use the Euler product

$$\zeta(2) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \prod_{p\ prime} (1 - \frac{1}{p^2})^{-1} = \prod_{p|q} (1 - \frac{1}{p^2})^{-1} \prod_{p\nmid q} (1 - \frac{1}{p^2})^{-1},$$

from which one derives (for odd n)

$$\frac{2^2}{\varphi(2)\psi(2)} \frac{n^2}{\varphi(n)\psi(n)} \sum_{\substack{i=1\\(i,2n)=1}}^{\infty} \frac{1}{i^2} = \zeta(2).$$

Now take an *even* n, then it follows from Lemma 4, Equation 47

$$|\mathscr{O}_n \cap s_1^{\pm}| = \varphi(n/2) \quad \text{and} \quad |\mathscr{O}_n \cap s_0^{\pm}| = \frac{1}{2}\varphi(n).$$

This gives Siegel-Veech constants

$$\begin{aligned} c_1^{\pm}(n) &= 4\frac{n^2}{\varphi(n)\psi(n)} \left[\frac{1}{2} \sum_{\substack{i=1\\(i,n/2)=1}}^{n/2} \frac{1}{i^2} + \frac{1}{4} \sum_{\substack{i=1\\(i,n)=1}}^{n/2} \frac{1}{(n/2-i)^2} \right] = \\ &= 4\frac{n^2}{\varphi(n)\psi(n)} \begin{cases} \frac{1}{2} \sum_{\substack{i=1\\(i,n/2)=1}}^{n/2} \frac{1}{i^2} + \frac{1}{4} \sum_{\substack{i=1\\(i,n)=1}}^{n/2} \frac{1}{(n/2-i)^2} & \text{if } 4 \mid n \\ \frac{1}{2} \sum_{\substack{i=1\\(i,n/2)=1}}^{n/2} \frac{1}{i^2} + \frac{1}{16} \sum_{\substack{i=1\\(i,n)=1}}^{n/2} \frac{4}{(n/2-i)^2} & \text{if } 4 \nmid n. \end{cases} \end{aligned} \tag{65}$$

Under the hypothesis $(i, n) = 1$ the difference $n/2-i$ can contain any prime-factor, but not the primes dividing $n/2$. Thus in case $4|n$ we can rearrange and find

$$c_1^{\pm}(n) = 3\frac{n^2/4}{\varphi(n/2)\psi(n/2)} \sum_{\substack{i=1\\(i,n/2)=1}}^{n/2} \frac{1}{i^2} \overset{m\to\infty}{\longrightarrow} 3 \cdot \zeta(2) \quad \text{if } 4 \mid n. \tag{66}$$

If $4 \nmid n$ (or $(n/2, 2) = 1$) we have to take into account that $2|(n/2 - i)$ and that $\frac{n^2}{\varphi(n)\psi(n)} = \frac{4}{3}\frac{n^2/4}{\varphi(n/2)\psi(n/2)}$ to find

$$c_1^{\pm}(n) = 4 \cdot \frac{4}{3} \cdot \frac{m^2}{\varphi(m)\psi(m)} \left[\frac{1}{2} \sum_{\substack{i=1\\(i,m)=1}}^{m} \frac{1}{i^2} + \frac{1}{16} \sum_{\substack{i=1\\(i,m)=1}}^{m/2} \frac{1}{i^2} \right] \overset{m\to\infty}{\longrightarrow} 3 \cdot \zeta(2) \tag{67}$$

using $m := n/2$ and Euler products again. This establishes the convergence for 2-symmetric torus covers. In the same manner one can show it for d-symmetric differentials for every d. We do not write down the details at this place and remark that a general convergence theorem, i.e. one which just depends on the Siegel-Veech formula and Ratner's Theorem, would be the desired and expected result.

Comments. The example treated in this paper as well as the various asymptotic formulas show that the asymptotic constants are very sensitive

invariants of the $\mathrm{SL}_2(\mathbb{Z})$ orbit of a particular translation surface in $\mathscr{F}_d^{sym}$. The previous discussion of saddle connections shows, that we can characterize cusps of a particular $\mathrm{SL}_2(\mathbb{Z})$-orbit using particular types of saddle connections. The asymptotic quadratic constants are good enough to distinguish (finite) $\mathrm{SL}_2(\mathbb{Z})$-orbits, moreover if one adds the horizontal cylinder decomposition of the modular fiber, the quadratic constants are good enough to distinguish cusps of a particular $\mathrm{SL}_2(\mathbb{Z})$ orbit.

As a matter of fact the quadratic growth rates of saddle conntections depend only on the saddle connections in the horizontal foliation of $\mathscr{F}_d^{sym}$. The generic constants however, obtained by approximation along the horizontal foliation are also the constants for saddle connections which will never be horizontal under application of an $A \in \mathrm{SL}_2(\mathbb{Z})$.

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